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THE

ELEMENTS

OF

TRIGONOMETRY.

CONTAINING,

The Properties, Relations, and Calculations of
SINES, TANGENTS, SECANTS, &c.

The Doctrine of the SPHERE, and the PRINCIPLES of PLAIN and SPHERICAL TRIGONOMETRY.

All plainly and clearly demonſtrated.

Emerson, William, 1701-1782

Quicquid in Aſtronomicis, Geographicis, vel Nauticis efficiendum, Trigonometriæ attribuendum eſt.

UNIV. OF MICH.

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THE PREFACE.

THE *Uſe of* Trigonometry *is ſo great in all the Parts of the Mathematics, that He muſt have made a very little Progreſs therein, who is not ſenſible of it. Its help is call'd in upon every Occaſion, and its great Service is clearly apparent in Calculations of all Sorts, both upon the Earth, upon the Seas, and in the Heavens. By this, the Diſtances of Objects upon the Earth may be certainly known, if they can but be ſeen; tho' we cannot come near to meaſure them: Likewiſe the geographical Diſtances of Places on the Earth; and their ſeveral Poſitions to one another. Navigation depends entirely upon it. Surveying and Dialling owe their greateſt Exactneſs to it. It is of ſingular Service in military Affairs: And* Mars, *without this, might live peaceably at Home. Upon the Wings of Trigonometry (as* Plato *ſays), we mount up from the Earth to the Heavens, meaſure the Diſtances of all the Stars, and range them in their proper Order. And without it* Urania'*s Sons may throw aſide their Inſtruments, their Books and Tables; or rather, without this, they would never have had any ſuch thing; and conſequently Mankind had remained utterly ignorant of this moſt beautiful Syſtem of the World. In ſhort, the Art of Trigonometry is of ſuch univerſal Uſe and Extent, that it would be an endleſs Taſk to enumerate all the various purpoſes to which it is ſubſervient; and the moſt important Branches of Knowledge would be loſt and uſeleſs if we wanted it.*

This makes me wonder that no body has given us an entire Syſtem of this Art; which ſhall contain all the Principles, *as well as the Rules of* Practice. *Inſtead of that, moſt Authors content themſelves with little more than explaining the common Caſes of ſolving Triangles, and fill up their Books with heaps of Examples, to ſhew the Uſe and Application thereof: Whilſt they dip very little into the Theory, and leave the Principles thereof very much in the Dark.*

In this Treatiſe I have adventured to lay down the Whole both Theory *and* Practice; *and to take in all Things of any Conſequence that any way belong to the Subject. And here I account all theſe Properties of Triangles or other Figures that have any Relation to the meaſuring their Angles by Degrees, or by Sines, and Tangents,* &c. *to belong to* Trigonometry, *as their proper Subject: referring their other Properties to* Geometry, *which have no ſuch Regard to the Meaſures of their Angles.*

In purſuance of this Deſign, I have in the firſt Book laid down in a few Propoſitions, the Relations of Sines, Coſines, &c, *of Arches; likewiſe of the double Arches, and of the Sums and Differences of Arches. Among which there are a few Propoſitions, ſo extenſive and general, as to comprehend a great Number of particular Propoſitions, evident only by Inſpection.*

Then I have given the Method of calculating natural *and* artificial Sines, Tangents, &c. *of any Arch: And from thence thoſe of their* multiple Arches. *And here I have been obliged to have Recourſe to the Method of* Fluxions, *and the Method of* Increments; *there being no other poſſible way to effect it. Which if any of my Readers happen not to underſtand, they may paſs by the Inveſtigations, and make equal Uſe of the Concluſions, nevertheleſs.*

Then you have a few general Propoſitions concerning angular Sections, Inſcription *of* Polygons, *and the* Properties of Chords *inſcrib'd in a Circle.*

In the ſecond Book you have plain Trigonometry *with the Solution of all the* Caſes, *ſeveral ways. Right angled Triangles are reſolv'd arithmetically, logarithmically, and algebraically: And oblique ones logarithmically and algebraically. By the firſt Method all the Caſes of right angled Triangles are reſolved by common Arithmetic, without any Books or Tables whatever. And this comes very near the Truth, and is ſufficiently exact for common Uſes: And is no further burthenſome to the Memory than getting by heart two or three Proportions, with certain fixt Numbers; and the* 47. I. *Euclid. And the ſame way may oblique Triangles be reſolved, by letting fall a Perpendicular.*

In the third Book you have all that is material in the Doctrine *of the* Sphere; *likewiſe the Properties of* Spherical Triangles; *and the Principles of* Spherical Trigonometry; *in which are ſome compound or general Propoſitions, including ſeveral particular ones. Among theſe there are ſome that I have taken the Liberty to demonſtrate algebraically, to avoid a more prolix Method of Demonſtration. Then follows the Solution of all the Caſes of ſpherical Triangles; in oblique ones both logarithmically and algebraically.*

In the firſt Sect. of the firſt Book, I have inſerted ſeveral Scholia *here and there: The uſe of which to Algebraiſts, will be too evident to be further inſiſted on. And for the ſame Reaſon it was, that I inſerted the Solution of the Caſes algebraically.*

All theſe Things being laid down here in this Book, in the ſhorteſt Manner poſſible; I think I may venture to entitle it The Elements of Trigonometry. *I won't* *ſay that I have quite exhauſted the Subject, but I am of Opinion that I have omitted little or nothing of any Conſequence.*

It would be needleſs giving any Examples in Numbers, as likewiſe the Application of Trigonometry to any Part of Mathematicks: That being very eaſy for the Reader to do himſelf; eſpecially as all Authors abound in theſe Things.

If what I have done meet with general Approbation, I have my End; if not I can only ſay,

In magnis voluiſſe ſat eſt.

W. Emerſon.

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Characters uſed in this Book.

+ Addition.
− Subtraction of the following Quantity.
× Multiply'd by.
⊥ Perpendicular to.
∥ Parallel to.
:: Proportion.
> Greater than.
< Leſſer than
∠ Angle.
∽ The Difference.
S. Sine; Coſ. Coſine; T. Tangent; Cot. Cotangent; Sec. Secant; Coſec. Coſecant; verſ. verſed ſine; Sup. Supplement; Seg. Segments; diff. difference.

THE ELEMENTS OF TRIGONOMETRY.

BOOK I.

The Properties of Sines, Tangents, Secants, &c. *of Arches; the Calculation of them, both natural, and artificial; and alſo thoſe of multiple Arches; angular Sections,* &c.

DEFINITIONS.

FIG. 1.

1. AN *Arch* of a Circle is any Part of its Circumference as A B. A Quarter of the Circumference is called a Quadrant as A D; and the half a Semi-circle as A D I.

2. The *Radius* is a Line drawn from the Center to the Circumference, as C A.

3. The *Complement* of an Arch is what it wants of a Quadrant, B D is the Complement of A B.

4. The *Supplement* is what it wants of a Semi-circle, as B I is the Supplement of A B.

Cor. 1. The Difference of two Arches is $\hat{=}$ Difference of their Supplements.

Cor. 2. The Complement of half an Arch is equal to half the Supplement of the whole Arch. For let Q be a Quadrant and A an Arch; then $Q-\frac{1}{2}A$ is the Complement of half the Arch, and $2Q-A$ is

FIG. 1. the Supplement of the Whole, and $Q-\frac{1}{2}A$ is half the Supplement of the Whole.

Cor. 3. The Difference of an Arch and its Complement is equal to the Complement of twice the Arch. $A \backsim \overline{Q-A}$ is the Difference of the Arch and its Complement, and $2A \backsim Q$ is the Complement of twice the Arch, but $A \backsim \overline{Q-A} = 2A \backsim Q$.

5. The *Chord* or *Subtenſe* of an Arch AB, is a right Line AB draw between the extream Points of the Arch.

6. The *Sine* (or right Sine) of an Arch AB, is a right Line BF drawn from one End of the Arch perpendicular to the Diameter paſſing thro' the other End of it.

Cor. The Sine of an Arch is half the Cord of the double Arch.

7. The *Coſine* of an Arch is the Diſtance between the Center and the Sine, as CF.

8. The *verſed Sine* of an Arch AB, is the Part of the Diameter, between the Beginning of the Arch and the Sine, as AF.

Cor. The verſed Sine + Coſine = Radius.

9. The *Coverſed Sine* is the Line DE, between the End of the Quadrant AD, and BE perpendicular to DC.

10. The *Tangent* of an Arch BA is the Line AG drawn from one End A of the Arch perpendicular to the Diameter, till it cuts the Line CBG drawn thro' the other End of the Arch.

11. The *Cotangent* of an Arch AB is the Line DH drawn perpendicular to DC at the End of the Quadrant AD, till it meet the Line CBH drawn thro' the Top of the Arch AB.

12. The *Semi-tangent* of an Arch AB, is the Line KC intercepted between the Center and the Line IK paſſing thro' the End B of the Arch.

Cor. The *Semi-tangent* of an Arch is = tangent of half the Arch. For if $AT=TB$, then $\angle ACT = \angle AIB$. Whence $CR=AR$.

13. The *Secant* of an Arch A B is the Line C G drawn from the Center of the Circle, thro' the Top of the Arch A B, till it meets the Tangent A G. FIG. 1.

14. The *Coſecant* of an Arch A B is the Line C H drawn from the Center, thro' the End B of the Arch, till it meets the Co-tangent D H in H.

Cor. 1. Hence the Coſine, Cotangent, Coſecant, coverſed Sine of an Arch, is equal to the Sine, Tangent, Secant, verſed Sine, of its Complement reſpectively.

For the Sine E B of the Arch D B (the Complement of A B) is = C F the Coſine of A B. Alſo D H the Cotangent of A B, is the Tangent of its Complement D B. Alſo C H the Coſecant of A B = Secant of its Complement D B. And laſtly D E the coverſed Sine of A B is the verſed Sine of its Complement D B.

Cor. 2. A Sine, Tangent and Secant, Coſine, Cotangent, Coſecant, and coverſed Sine are common to two Arches which are the Supplements of each other. For theſe Lines belong as well to the Arch I B, as to A B.

15. A *Degree* is the 360th Part of the Circumference of every Circle; and therefore a Semi-cricle contains 180 Degrees, and a Quadrant 90°. Alſo the 60th Part of a Degree is called a Minute, the 60th Part of a Minute a Second, *&c.*

16. An *Angle* A C B is the Inclination of two Lines A C, C B; and the Meaſure of that Angle is the Degrees in the Arch A B.

Note, The middle Letter ſtands at the Angle.

SCHOLIUM.

Tho' the Coſine, Tangent, Cotangent and Secant are common to two Arches which are the Supplements to each other; yet in an Arch greater than a Quadrant, theſe Lines are truly negative, and muſt be ſo eſteemed, becauſe they are drawn the contrary way to thoſe in an Arch leſs than a Quadrant. And there-

FIG. 1. therefore the Sine, coverſed Sine, and Coſecant, are always affirmative in all Arches leſs than a Semicircle, but negative in greater Arches. And in general the Sines of all Arches in the firſt and ſecond Quadrants are affirmative, in the third and fourth negative. The Coſines in the firſt and fourth Quadrants are affirmative, in the ſecond and third negative. The Tangents in the firſt and third Quadrants are affirmative, in the ſecond and fourth negative. The Cotangents in the firſt and third are affirmative, in the ſecond and fourth negative. The Secants in the firſt and fourth are affirmative, in the ſecond and third negative. The Coſecants in the firſt and ſecond are affirmative, in the third and fourth are negative. All verſed Sines are affirmative: Moreover Sines, Tangents, *&c.* of negative Arches are contrary in their Signs to thoſe of affirmative Arches.

FIG. I.

SECT. I.

The Relation of the Sines, Tangents and Secants of Arches.

PROP. I.

In any Arch, theſe are reſpectively porportional,

Radius : *Sine* : *Coſine* : :
Secant : *Tangent* : *Radius* : :
Coſecant : *Radius* : *Cotangent.*

The Meaning of the Propoſition is, that any four Quantities lying in Form of a Rectangle are proportional. As thus; Rad : Sine : : Secant : Tangent. Or Rad : Coſine : : Secant : Radius. Or Tangent : Radius : : Radius : : Co-tangent. Or Cotangent : Coſine : : Coſecant : Radius. And ſo of others. The ſame is to be underſtood of the following Propoſitions. Likewiſe if you have ſeveral Quantities placed thus, A : B : C : : D : E : F, the meaning is, that A : B : : D : E, or A : C : : D : F, or B : C : : E : F, *&c.* or laſtly if A : B : : C : D : : E : F, *&c.* it ſignifies that A : B : : C : D, or A : B : : E : F, or C : D : E : F, and ſo of others.

For in the Similar-triangles BFC, GAC, CDH; BC : BF : CF : : GC : GA : AC : : CH : CD : DH.

Cor. 1. Radius Square = Sine Square + Coſine Square = Secant Square — Tangent Square = Coſecant Square — Cotangent Square. For the Triangles B F C, G A C, C D H are right angled.

Cor. 2. Radius Square = Tangent × Cotangent = Coſine × Secant = Sine × Coſecant.

Cor. 3. In any Arches, the Sine is as the Coſine × Tangent, or as Tangent divided by Secant, or as Coſine divided by Cotangent, or reciprocally as the Coſecant.

Cor. 4. Tangent is as the Sine × Secant, or as

FIG. 1 $\frac{\textit{Sine}}{\textit{Coſine}}$, or as $\frac{\textit{Secant}}{\textit{Coſecant}}$, or reciprocally as the Cotangent.

Cor. 5. Secant is as Tangent × Coſecant, or as $\frac{\textit{Tangent}}{\textit{Sine}}$, or as $\frac{\textit{Coſecant}}{\textit{Cotangent}}$, or reciprocally as the Coſine.

Cor. 6. Coſine is reciprocally as the Secant, Cotangent reciprocally as the Tangent, Coſecant reciprocally as the Sine.

Cor. 7. The Sum of the Squares of the Sine and verſed Sine of an Arch = 4 times the Square of the Sine of half the Arch. For A B = 2 A L, and the Triangle A B F is right angled.

Cor. 8. The Cord of an Arch is a mean Proportional between the verſed Sine and Diameter. For A F, A B, A I are continually Proportional.

SCHOLIUM.

Let A=any Arch.
r =Radius.
s =Sine.
c =Coſine.
t=Tangent
τ=Cotangent
ſ=Secant
σ =Coſecant.
v=verſed Sine.
V=verſ.S.Sup.

Then

$$\text{I. } s=\sqrt{rr-cc}=\frac{ct}{r}=\frac{tr}{\text{ſ}}=\frac{tr}{\sqrt{rr+tt}}=\frac{rc}{\tau}=\frac{rr}{\sqrt{rr+\tau\tau}}$$
$$=\frac{rr}{\sigma}=\frac{\text{ſ}c}{\sigma}=\frac{t\tau}{\sigma}=\frac{r}{\text{ſ}}\sqrt{\text{ſſ}-rr}=\sqrt{\text{V}v}=\sqrt{2rv-vv}.$$

$$\text{II. } c=r-v=\text{V}-r=\sqrt{rr-ss}=\frac{rs}{t}=\frac{rr}{\sqrt{rr+tt}}=\frac{rr}{\text{ſ}}$$
$$=\frac{t\tau}{\text{ſ}}=\frac{s\sigma}{\text{ſ}}=\frac{s\tau}{r}=\frac{r\tau}{\sigma}=\frac{r\tau}{\sqrt{rr+\tau\tau}}=\frac{r}{\sigma}\sqrt{\sigma^2-rr}.$$

$$\text{III. } t=\frac{rs}{c}=\frac{rs}{\sqrt{rr-ss}}=\frac{r}{c}\sqrt{rr-cc}=\frac{rr}{\tau}=\frac{rr}{\sqrt{\sigma^2-r^2}}=$$
$$\frac{c\text{ſ}}{\tau}=\sqrt{\text{ſ}^2-rr}=\frac{s\sigma}{\tau}=\frac{r\sqrt{2rv-vv}}{r-v}=\frac{r\sqrt{2r\text{V}-\text{V}^2}}{\text{V}-r}.$$

IV. $\tau=$

FIG. 1.

$$\text{IV. } \tau=\sqrt{\sigma\sigma-rr}=\frac{rr}{t}=\frac{s\sigma}{t}=\frac{c\text{ſ}}{t}=\frac{rc}{s}=\frac{r}{s}\sqrt{rr-ss}=$$

$$\frac{rc}{\sqrt{rr-cc}}=\frac{rr}{\sqrt{\text{ſſ}-rr}}=\frac{r-v}{\sqrt{2rv-vv}}r=\frac{\mathrm{V}-r}{\sqrt{2r\mathrm{V}-\mathrm{V}^2}}r.$$

$$\text{V. } \text{ſ}=\sqrt{rr+tt}=\frac{rr}{c}=\frac{rt}{s}=\frac{r^3}{s\tau}=\frac{r\sigma}{\tau}=\frac{t\tau}{c}=\frac{s\sigma}{c}=$$

$$\frac{rr}{\sqrt{rr-ss}}=\frac{r}{\tau}\sqrt{rr+\tau\tau}=\frac{r\sigma}{\sqrt{\sigma\sigma-rr}}=\frac{rr}{r-v}=\frac{rr}{\mathrm{V}-r}.$$

$$\text{VI. } \sigma=\sqrt{rr+\tau\tau}=\frac{rr}{s}=\frac{r\text{ſ}}{t}=\frac{r\tau}{c}=\frac{r^3}{ct}=\frac{t\tau}{s}=\frac{c\text{ſ}}{s}=$$

$$\frac{rr}{\sqrt{rr-cc}}=\frac{r}{t}\sqrt{rr+tt}=\frac{r\text{ſ}}{\sqrt{\text{ſſ}-rr}}=\frac{rr}{\sqrt{2rv-vv}}.$$

$$\text{VII. } v=2r-\mathrm{V}=r-c=r-\sqrt{rr-ss}=r-\frac{rr}{\sqrt{rr+tt}}=$$

$$r-\frac{r\tau}{\sqrt{rr+\tau\tau}}=r-\frac{rr}{\text{ſ}}=r-\frac{r}{\sigma}\sqrt{\sigma\sigma-rr}.$$

$$\text{VIII. } \mathrm{V}=2r-v=r+\sqrt{rr-ss}=r+c=r+\frac{rr}{\sqrt{rr+tt}}=$$

$$r+\frac{r\tau}{\sqrt{r^2+\tau^2}}=r+\frac{r}{\sigma}\sqrt{\sigma^2-rr}=\frac{s+t}{t}r=r+\frac{s\tau}{r}=r+\frac{rr}{\text{ſ}}=$$

$$\frac{\tau+\sigma}{\sigma}r.$$

The coverſed Sine is eaſily found expreſs'd by any of the reſt, by only ſubtracting the Sine from Radius.

PROP. II.

In any Arch theſe Lines are reſpectively proportional.

Radius : *Sine of an Arch* : *Coſine Arch* : :
2 *Sine Arch* : *Verſ.* 2 *Arch* : *Sine* 2 *Arch* : :
2 *Coſine Arch* : *Sine* 2 *Arch* : *Verſ. Sup.* 2 *Arch.*

For the Triangles CAL, BAF, IBF are ſimilar, and

FIG. 1. and $IB = 2CL$, therefore $CA : AL : CL :: BA : AF : BF :: IB : BF : IF$.

Cor. 1. From hence and Prop. I. it is evident, that theſe are reſpectively proportional.

Radius : *Sine of an Arch* : *Coſine Arch* ::
Secant Arch : *Tangent Arch* : *Radius* ::
Coſecant Arch : *Radius* : *Cotangent Arch* ::
2 *Sine Arch* : *Verſ.* 2 *Arch* : *Sine* 2 *Arch* ::
2 *Coſine Arch* : *Sine* 2 *Arch* : *Verſ. Sup.* 2 *Arch.*

Cor. 2. In any Arch, $\frac{2\ \textit{Sine Square}}{\textit{Radius}}$ = verſed Sine of the double Arch, that is $\frac{2ss}{r} =$ verſ. 2 A. See Sch. to Pr. I.

Cor. 3. In any Arch, $\frac{2\ \textit{Sine} \times \textit{Coſine}}{\textit{Radius}}$ = Sine of the double Arch, $\frac{2cs}{r} =$ S. 2 A.

Cor. 4. In any Arch, $\frac{2\ \textit{Coſine Square} - \textit{Radius Square}}{\textit{Radius}}$ = Coſine of the double Arch. For Coſ. 2 A $= r -$ verſ. 2 A $= r - \frac{2ss}{r} = \frac{rr - 2ss}{r} = \frac{2cc - rr}{r}$.

Cor. 5. In any Arch, $\frac{2\ \textit{Radius Square}}{\textit{Cotangent} - \textit{Tangent}}$ = Tangent of twice the Arch. For Coſ. : Sine :: Rad. : Tangent, that is by Cor. 4, 3. $\frac{2cc - rr}{r}$ or $\frac{cc - ss}{r} : \frac{2cs}{r} :: r : \frac{2csrr}{cc - ss . r} = \frac{2rr}{\frac{rc}{s} - \frac{rs}{c}} =$ (by Schol. Pr. I.) $\frac{2rr}{\tau - t}$.

Cor. 6. In any Arch, ½ Cotangent — ½ Tangent = Cotangent of the double Arch. This follows from Cor. 5, becauſe Radius Square divided by the Tangent is = Cotangent.

Cor.

FIG. I.

Cor. 7. In any Arch, $\frac{\textit{Secant} \times \textit{Radius}}{2\ \textit{Coſine} - \textit{Secant}}$ = Secant of the double Arch. For $\frac{\textit{Radius Square}}{\textit{Coſine}}$ = Secant, that is by Cor. 4. $\frac{rr}{2cc - rr}r = \frac{\frac{rr}{c}}{2c - \frac{rr}{c}}r =$ (by Sch. Pr. I.) $\frac{rs}{2c - s}$ = Secant of 2 Arch.

Cor. 8. In any Arch, $\frac{\textit{Secant} \times \textit{Coſecant}}{2\ \textit{Radius}}$ = Coſecant of the double Arch. For by Cor. 1. of this Prop. $\frac{2rs}{s}$ = S. 2 Arch, and (by Sch. Pr. I.) $\frac{\textit{Radius Square}}{\textit{Sine}}$ = Coſecant, that is $\frac{rrs}{2rs} = \frac{s\sigma}{2r}$, becauſe $\frac{rr}{s} = \sigma$, by Schol. Pr. I.

Cor. 9. In any Arch, $\frac{8\ \textit{Sine Square} \times \textit{Coſine Square}}{\textit{Radius Cube}}$ = verſed Sine of the Quadruple Arch. For, by Cor. 3. $\frac{2cs}{r}$ = S. 2 A, and by Cor. 2. $\frac{2 \times 4ccss}{r \times rr}$ = verſed Sine of twice 2 A.

Cor. 10. Hence alſo, the verſed Sines of two Arches are as the Squares of the Sines of their half Arches.

Cor. 11. The Sines of two Arches are as the Rectangles of the Sine and Coſine of their half Arches.

Cor. 12. The Coſecants of two Arches are as the Rectangles of the Secant and Coſecant of their half Arches.

Cor. 13. The verſed Sine of the Sup. of an Arch : ½ Radius : : Sine Square : Sine Square of ½ the Arch.

For verſ. Sup. 2A : Coſ, A : : S.2A : S.A.
And 2Coſ. A : Rad. : : S.2A : S.A. whence
by multiplying, verſ. Sup. 2A : ½Rad. : : S^2.2A : S^2.A.

Cor.

FIG. 1. *Cor.* 14. What has here been demonſtrated of the Relation of Sines and Coſines to one another, holds equally true, in reſpect of the Chords of Arches, and their Supplements; puting the Diameter inſtead of Radius.

Cor. 15. Hence alſo it follows from Cor. 6. that the Difference between the Tangent and Cotangent of an Arch = twice the Tangent of the Difference between that Arch and its Complement.

For the Difference between an Arch and its Complement is equal to the Complement of twice that Arch. Hence alſo

Cor. 16. Let A be any Arch leſs than 45°, then Tangent of $\overline{45+A}^{\circ}$ = Tangent of $\overline{45-A}^{\circ}+2$ Tangents of 2A. For 2A is the Difference of the Arches $\overline{45+A}$ and $\overline{45-A}$.

SCHOLIUM.

Hence alſo the Sines, Coſines, &c. of the double Arches may be expreſſed more univerſally thus, (See Sch. Pr. I.)

I $\frac{2cs}{r}=\frac{2ss}{t}=\frac{2cc}{\tau}=\frac{2rs}{\text{ſ}}=\frac{2rc}{\sigma}=\frac{2rrt}{rr+tt=\text{ſſ}}=\frac{2rr}{\tau+t}$ $=\frac{2r^2\tau}{rr+\tau\tau=\sigma\sigma}=$ *Sine of 2 Arch.*

II. $\frac{cc-ss}{r}=\frac{rr-2ss}{r}=\frac{2cc-rr}{r}=\frac{rr-tt}{rr+tt}r=\frac{\tau-t}{\tau+t}r$ $=\frac{\tau\tau-rr}{\tau\tau+rr}r=\frac{2rr-\text{ſſ}}{\text{ſſ}}r=\frac{2c-\text{ſ}}{\text{ſ}}r=\frac{\sigma^2-2r^2}{\sigma^2}r=\frac{\sigma-2s}{\sigma}r$ $=$ *Coſine of 2 Arch.*

III. $\frac{2rrt}{rr-tt}=\frac{2rr}{\tau-t}=\frac{2rcs}{rr-2ss}=\frac{2rcs}{2cc-rr}=\frac{2rr\tau}{\tau\tau-rr}$ $=$ *Tangent of 2 Arch.*

IV. $\frac{rr-tt}{2t}=\frac{\tau\tau-rr}{2\tau}=\frac{\tau-t}{2}=\frac{rr-2ss}{2cs}r=\frac{2cc-rr}{2cs}r$ $=$ *Co-tangent of 2 Arch.*

FIG.

$$\text{V. }\frac{ſſr}{2rr-ſſ}=\frac{ſr}{2c-ſ}=\frac{r^3}{2cc-rr}=\frac{r^3}{rr-2ss}=\frac{rr+tt}{rr-tt}r$$
$$=\frac{\tau+t}{\tau-t}r=\frac{\tau\tau+rr}{\tau\tau-rr}r=\textit{Secant of 2 Arch.}$$

$$\text{VI. }\frac{ſ\sigma}{2r}=\frac{rſ}{2s}=\frac{r\sigma}{2c}=\frac{r^3}{2cs}=\frac{rr+tt=ſſ}{2t}=\frac{\tau+t}{2}=$$
$$\frac{rr+\tau\tau=\sigma\sigma}{2\tau}=\textit{Cosecant of 2 Arch.}$$

$$\text{VII. }\frac{2ss}{r}=\frac{2rr-2cc}{r}=\frac{2vV}{r}=\frac{2v\times 2\overline{r-v}}{r}=\frac{2rtt}{rr+tt=ſſ}$$
$$=\frac{2rt}{\tau+t}=\frac{2r^3}{rr+\tau\tau=\sigma\sigma}=\frac{2rs}{\sigma}=\frac{ſſ-rr}{ſſ}\times 2r=\frac{ſ-c}{ſ}\times 2r$$
$$=\textit{versed sine of 2 Arch.}$$

$$\text{VIII. }\frac{2rr-2ss}{r}=\frac{2cc}{r}=\frac{2rr}{rr+tt}r=\frac{2\tau\tau}{rr+\tau\tau}r=\frac{2r\tau}{t+\tau}=$$
$$\frac{2r^3}{ſſ}=\frac{2cr}{ſ}=\frac{2\sigma^2-2r^2}{\sigma^2}r=\frac{2\sigma-2s}{\sigma}r=\frac{\overline{V-r}|^2}{\frac{1}{2}r}=\frac{\overline{r-v}|^2}{\frac{1}{2}r}=$$
$$\textit{verſ. Sup. } 2\text{A.}$$

The coverſed Sine may be expreſs'd by any of the Reſt, by ſubtracting the Value of the Sine from Radius. Alſo other Values of each Quantity may be inſerted by Sch. Pr. I.

PROP. III.

If three Arches AF, AN, AP *be in arithmetic Progreſſion, then theſe are reſpectively proportional.* 2.

Radius : *S. mean Arch* : *Coſ. mean Arch* ::
2 *S. mean Arch* : *Verſ.* 2 *mean Arch.* : *S.* 2 *mean Arch* ::
2 *Coſ. mean Arch.* : *S.* 2 *mean Arch.* : *Verſ.Sup.* 2 *mean Arch* ::
2 *S. common Diff.* : *Diff.Coſ.Extreams* : *Diff. Sin. Extreams* ::
2 *Coſ. com. Diff.* : *Sum.Sin.Extreams* : *Sum Coſ. Extreams.*

Note, *Diff. Sines* = *S. greater Arch* — *S. leſſer Arch. Diff. Coſines* = *Coſ. leſſer* — *Coſ. greater Arch* = *Diff. verſed Sines* = *verſ. greater* — *verſ. leſſer Arch, univerſally.*

Draw PCM and PF and MF; and AD ∥ FP, and FS ∥ AB, and CE, MS ⊥ AB. Then MS = PG +

FIG. 2. $PG+FH$, and $SF=CG+CH$. And Arch $OM=PN=NF$. Therefore $MF \parallel NC$, and $MF=2CR$. Whence the Triangles CAQ, DAK, BDK, PFI, FMS are ſimilar; and $CA:AQ:CQ::DA:AK:DK::BD:DK:BK::PF:FI:PI::FM:MS:FS$. *Q. E. D.*

Cor. 1. Since the mean $=\frac{1}{2}$ Sum of the Extreams, therefore in any two Arches,

Radius	: S. $\frac{1}{2}$ Sum	: Coſ. $\frac{1}{2}$ Sum	::
2 S. $\frac{1}{2}$ Sum	: Verſ. Sum	: S. Sum	::
2 Coſ. $\frac{1}{2}$ Sum	: S. Sum	: Verſ. Sup. Sum	::
2 S. $\frac{1}{2}$ Difference	: Diff. Coſines	: Diff. Sines	::
2 Coſ. $\frac{1}{2}$ Diff.	: Sum Sines	: Sum Coſines.	

Cor. 2. Let A, E be two Arches, A the greater, E the leſſer, then

$\frac{1}{2}$ Rad.	: S. A	: Coſ. A	::
S. E	: Coſ. $\overline{A-E}$ — Coſ. $\overline{A+E}$	: S. $\overline{A+E}$ — S. $\overline{A-E}$	::
Coſ. E	: S. $\overline{A+E}$ + S. $\overline{A-E}$	: Coſ. $\overline{A-E}$ + Coſ. $\overline{A+E}$.	

This will appear by Cor. 1. puting $A=$ half the Sum of 2 Arches, and $E=$ half their Difference. For then the Arches will be $\overline{A+E}$, and $\overline{A-E}$.

Cor. 3. As $\frac{1}{2}$ Rad. : S. $\frac{1}{2}$ Sum of two Arches :: S. $\frac{1}{2}$ Difference : Difference of their Coſines, or of their verſed Sines.

Cor. 4. Rectangle of Radius and the Difference of the verſed Sines (or Coſines) of two Arches = twice the Rectangle of the Sine of half the Sum, and Sine of half the Difference of theſe Arches.

Cor. 5. If three Arches AF, AN, AP are in arithmetic Progreſſion, the Rectangle of the Sine of the mean Arch, and verſed Sine of the common Difference = Rectangle of Radius, and the Difference between the Sine of the mean Arch, and half the Sum of the Sines of the extream Arches.

For $CA \times \overline{PG+FH} = 2CR \times AQ = 2AQ \times \overline{CN-NR}$, whence $2AQ \times NR = 2AQ \times CA - CA \times \overline{PG+FH}$. Hence,

Cor.

FIG. 2.

Cor. 6. The ſecond Difference of the Sines FH, AQ, PG, $\times$ Radius $=$ twice the Sine of the mean Arch $\times$ verſed Sine of the common Difference. For the ſecond Difference $= 2\,AQ - FH - PG$.

Cor. 7. As Radius :
To twice the Coſine of any Arch : :
So the Sine of n times that Arch :
To Sum of the Sines of $n-1$ and $n+1$ times that Arch : :
And ſo the Coſine of n times that Arch :
To the Sum of the Coſines of $n-1$ and $n+1$ times that Arch.

For if A be the Arch, then $\overline{n-1}.A$, nA, $\overline{n+1}.A$ are in arithmetic Progreſſion, and A the common Difference.

Cor. 8. Let A be any Arch leſs than 30°, then Sine of $A \times \sqrt{3}$ + Sine of $\overline{30-A} =$ Sine of $\overline{30+A}$.

For the Arches $30-A$, 30°, and $30+A$ are in arithmetic Progreſſion, and Coſ. mean $30 = \frac{1}{2}\sqrt{3} \times$ Radius. And by this Prop. Rad. : Coſ. 30° : : 2 Sine of A : S. $\overline{30+A}$ — S. $\overline{30-A}$.

Cor. 9. If A be any Arch leſs than 30, then S. $\overline{30+A}$ + S. $\overline{30-A}$ = S. $\overline{90-A}$.

For the S. 30 the mean $= \frac{1}{2}$ Rad. and by this Prop. Rad. : $\frac{1}{2}$ Rad. : : 2 Coſ. A : S. $\overline{30+A}$ + S. $\overline{30-A}$ = Coſ. A.

Cor. 10. If A be any Arch leſs than 30°, then S. $\overline{60+A} \times \sqrt{3}$ — S. $\overline{30+A}$ = S. $\overline{90-A}$.

For if the Arches be A, $60+A$, $120+A$, and Radius $\times \frac{1}{2}\sqrt{3}$ = S. 60 the common Difference. Then Rad. : Rad. $\times \sqrt{3}$: : S. $\overline{60+A}$: Coſ. A — Coſ. $\overline{120+A}$ = Coſ. A + Coſ. $\overline{60-A}$, that is $1 : \sqrt{3}$: : S. $\overline{60+A}$: S. $\overline{90-A}$ + S. $\overline{30+A}$.

Cor. 11. If A be any Arch leſs than 45°; then S. $A \times \sqrt{2}$ + S. $\overline{45-A}$ = S. $\overline{45+A}$. For in the Arches $45-A$, 45, $45+A$; $\frac{1}{2}\sqrt{2} \times$ Rad. = Coſ. the mean, 45.

Cor. 12. Let A be any Arch leſs than 60; then

S. A+

FIG. $S.\ A + S.\ \overline{60 - A} = S.\ \overline{60 + A}$. For in the Arches $60 - A$, 60, $60 + A$, the Coſ. mean, $60^{\circ} = \frac{1}{2}$ Rad.

Cor. 13. What is demonſtrated in this Prop. and Cor. 2, 3, 7, of Sines and Coſines, holds of Cords of Arches, and their Supplements, puting Diameter for Radius.

PROP. IV.

In any two Arches, theſe are reſpectively proportional.

$\frac{1}{2}$ Radius :	*S. $\frac{1}{2}$ Sum* :	*Coſ. $\frac{1}{2}$ Sum* :	*S. $\frac{1}{2}$ Diff.* :	*Coſ. $\frac{1}{2}$ Diff.*	: :
S. $\frac{1}{2}$ Sum :	*Verſ. Sum* :	*S. Sum* :	*Diff. Coſ.* :	*Sum Sines*	: :
Coſ. $\frac{1}{2}$ Sum :	*S. Sum* :	*V. Sup. Sum* :	*Diff. Sines* :	*Sum Coſ.*	: :
S. $\frac{1}{2}$ Diff. :	*Diff. Coſ.* :	*Diff. Sines* :	*Verſ. Diff.* :	*S. Diff.*	: :
Coſ. $\frac{1}{2}$ Diff. :	*Sum Sines* :	*Sum Coſ.* :	*S. Diff.* :	*V. Sup. Diff.*	

Note, *Diff. Sines and Coſines, is the ſame as in Pr. III.*

The 3 firſt perpendicular Rows are evident from the laſt Prop. and the firſt, fourth, and fifth Terms in the 2 laſt perpendicular Rows follow from Prop. II. and the remaining Terms are fill'd up by the Proportions in this very Prop. thus, $\frac{1}{2}$ Rad. : S. $\frac{1}{2}$ Diff. : : S. $\frac{1}{2}$ Sum : Diff. Coſines, which therefore will ſtand in the ſecond Place of the fourth perpendicular Column, as well as in the fourth Place of the ſecond Column, &c.

Hence ſeveral Corollaries follow of their own accord; as

Cor. 1. As verſ. Sup. Sum of 2 Arches : Diff. Sines : : Diff. Sines : verſed Sine of their Difference.

Cor. 2. Sine of the Sum of 2 Arches : Sum of their Sines : : Difference of their Sines : Sine of the Difference of the Arches.

Cor. 3. If three Arches are in arithmetical Progreſſion,

As the Sine of one Extream :

Sum of the Sines of the mean Arch, and common Difference : :

So their Difference (S. mean — S. com. Diff.) :

To the Sine of the other Extream.

For let $m =$ mean Arch, $d =$ com. Difference, and

then by Cor. 2. S. $\overline{m+d}$: S. m+S. d :: S. m—S. d : FIG.
S. $\overline{m-d}$.

Cor. 4. Let A be any Arch; m, n any Numbers, then
As Sine of $\overline{m-n}$. A :
S. mA+S. nA ::
S. mA—S. nA :
Sine of $\overline{m+n}$. A. By this Prop. hence

Cor. 5. As the Sine of an Arch :
Sum of the Sines of the double and ſingle Arch ::
So their Difference (S. double Arch — S. ſingle Arch) :
To Sine of the tripple Arch.

Cor. 6. As the Sine of an Arch :
Sum of the Sines of the triple and double ::
So their Difference (S. triple — S. double) :
To Sine of the Quintriple Arch, *&c.*

Cor. 7. Let $r=$ Radius, $s=$ Sine of the Arch 2 A, then the Sine of $\overline{45+A}=\sqrt{\frac{rr+rs}{2}}$, and the Sine of $\overline{45-A}=\sqrt{\frac{rr-rs}{2}}$.

For let $a=$S. $\overline{45+A}$. $e=$S. $\overline{45-A}$. Then by this Prop. as (S. Sum) r : (Sum Sines) $a+e$ :: (Diff. Sines) $a-e$: s the Sine of the Difference. Therefore

$$rs=aa-ee$$

but $rr=aa-ee$ by Cor. 1. Prop. I.

therefore $rr+rs=2aa$

and $rr-rs=2ee$.

Cor. 8. What has been demonſtrated in this Prop. and firſt 6 Corollaries concerning the Relation of Sines, and Coſines, holds equally true with reſpect to their Cords; puting the Cord of the Supplement inſtead of Coſine, and the Diameter inſtead of Radius.

SCHOLIUM.

Let there be two Arches A, E, and let

$C = \text{Coſ.}\ \overline{A+E}\ .\ c = \text{Coſ.}\ \overline{A-E}$

$V = \text{Verſ.}\ \overline{A+E}\ .\ v = \text{Verſ.}\ \overline{A-E}$

$Z = \text{Sine}\ \frac{A+E}{2}\ .\ z = \text{S.}\ \frac{A-E}{2}$

$Y = \text{Coſ.}\ \frac{A+E}{2}\ .\ y = \text{Coſ.}\ \frac{A-E}{2}$

$D = \text{Coſ.}\ \overline{2A+2E}\ .\ d = \text{Coſ.}\ \overline{2A-2E}$, Then

$$\text{S. } A \times \text{S. } E = \overline{Z+z} \times \overline{Z-z} = Z^2 - z^2 = \overline{y+Y} \times \overline{y-Y}$$
$$= y^2 - Y^2 = \frac{V-v}{2} r = \frac{c-C}{2} r.$$

$$\text{Coſ. } A \times \text{Coſ. } E = \frac{c+C}{2} r.$$

$$\text{Tan. } A \times \text{Tan. } E = \frac{c-C}{c+C} rr.$$

$$\text{S. } A \times \text{S. } E \times \text{Coſ. } A \times \text{Coſ. } E = \frac{cc-CC}{4} rr = \frac{d-D}{8} r^3.$$

$$\text{Coſ. } A \times \text{Coſ. } E \times \text{Tan. } A \times \text{Tan. } E = \frac{c-C}{2} r^3.$$

PROP. V.

3. *In any two Arches* (AD, DE), *the Sum of the Rectangles of the Sine of one into the Coſine of the other, is* = *Rectangle of the Radius, and the Sine of the Sum of theſe Arches.*

DEMONSTRATION.

Draw NF ∥ and NR ⊥ CA : then becauſe the Angles ENC, FNR are right, therefore ENF = CNR; and therefore the Triangles ENF, CNR and CDG are ſimilar; wherefore CD : DG :: CN : NR $= \frac{DG \times CN}{CD} =$ FI. And CD : CG :: EN :

EF =

FIG. 3.

$EF=\frac{CG\times EN}{CD}$, whence $EI=EF+FI=$ $\frac{CG\times EN+DG\times CN}{CD}$. *Q. E. D.*

Cor. 1. In any two Arches (A D, D E), the Rectangle of their Coſines — the Rectangle of their Sines = the Rectangle of Radius and the Coſine of their Sum.

For by ſimilar Triangles, $CD : CG :: CN : CR = \frac{CG\times CN}{CD}$. And $CD : DG :: (CN : NR ::) EN : NF=\frac{DG\times EN}{CD}=IR$. And $CI=CR-IR=$ $\frac{CG\times CN-DG\times EN}{CD}$.

Cor. 2. The Square of the Sine of the Sum of two Arches is = the Sum of the Squares of their Sines + twice their Rectangle multiply'd by the Coſine of the Sum and divided by Radius.

For let the Arches $A+E=S$, their Sines a, e, s; and c the Coſine of S. Then by this Prop. $s=\frac{a\sqrt{rr-ee}+e\sqrt{rr-aa}}{r}$ and $ss=aa+ee-\frac{2aaee}{rr}+\frac{2ae}{rr}\times\sqrt{rr-aa}\times\sqrt{rr-ee}$. But by Cor. 1. $\frac{\sqrt{rr-aa}\times\sqrt{rr-ee}}{r}-\frac{ae}{r}=c$. Therefore $ss=aa+ee+\frac{2c}{r}ae$.

Cor. 3. Hence if $A+E=90°$, then $aa+ee=rr$.
If $A+E=60$, then $aa+ee+ae=\frac{3}{4}rr$.
If $A+E=45$, then $aa+ee+ae\sqrt{2}=\frac{1}{2}rr$.
If $A+E=30$, then $aa+ee+ae\sqrt{3}=\frac{1}{4}rr$.

SCHOLIUM.

This Prop. and Cor. 1. holds univerſally for all Arches, puting negative Coſines for Arches greater than a Quadrant.

FIG. 3.

PROP. VI.

In two Arches (AD, AE), *the Rectangle of the Sine of the greater and Coſine of the leſſer — the Rectangle of the Sine of the leſſer and Coſine of the greater is = Rectangle of Radius and the Sine of the Difference of the Arches.*

DEMONSTRATION.

The Triangles CDG, CBI, and EBN are ſimilar; therefore $CG : DG :: CI : BI = \frac{CI \times DG}{CG}$. And $CD : CG :: EB$ or $EI - \frac{CI \times DG}{CG} : EN = \frac{CG \times EI - CI \times DG}{CD}$.

Cor. In any two Arches (AD, AE), the Rectangle of their Sines + the Rectangle of their Coſines is = Rectangle of the Radius and the Coſine of the Difference of the Arches.

For $BI = \frac{CI \times DG}{CG}$, and by ſimilar Triangles $CB = \frac{CI \times CD}{CG}$. And $CD : DG :: BE$ or $EI - \frac{DG \times CI}{CG} : BN = \frac{DG \times EI}{CD} - \frac{DG^2 \times CI}{CD \times CG} = \frac{DG \times EI}{CD} - \frac{CD^2 \times CI}{CD \times CG} + \frac{CG^2 \times CI}{CD \times CG} = \frac{DG \times EI}{CD} - CB + \frac{CG \times CI}{CD}$; therefore $\frac{DG \times EI + CG \times CI}{CD} = CB + BN = CN$.

SCHOLIUM I.

This Prop. and its Cor. holds univerſally for all Arches, putting negative Coſines for Arches greater than a Quadrant.

SCHO-

SCHOLIUM II.

From the foregoing Propoſitions it follows, that if Radius $=1$. And

$S, C=$ Sine, and Coſine of half the Sum of two Arches

$s, c=$ Sine, and Coſine of half their Difference, then

$Sc+Cs=$ Sine of the greater.
$Sc-Cs=$ Sine of the leſſer.
$Cc-Ss=$ Coſine of the greater.
$Cc+Ss=$ Coſine of the leſſer.

Alſo,

If $Z=$ Secant of the greater Arch, $z=$ Secant of the leſſer.

$Y=$ Coſecant of the greater, $y=$ Coſecant of the leſſer.

Then $\frac{YyZz}{Yy-Zz}=$ Secant of the Sum.

$\frac{YyZz}{Zy+Yz}=$ Coſecant of the Sum.

$\frac{YyZz}{Yy+Zz}=$ Secant of the Difference.

$\frac{YyZz}{Zy-Yz}=$ Coſecant of the Difference.

PROP. VII.

If three Arches (AF, AN, AP) *be in arithmetic Progreſſion,* 4.

The Sum of the Sines of the Extream Arches :
To their Difference (*S. greater — S, leſſer*) ::
As Tang. mean Arch :
To Tang. common Difference of the Arches.

For drawing the Lines as in the Figure, then by ſimilar Triangles, 2 OB or PG+FH : 2 OD or PI :: OB : OD :: OL : OF :: NM : NK.

Cor. 1. In any two Arches (AF, AP;)

As Sum of their Sines :
To their Difference (*S. greater — S. leſſer*) ::

FIG. 4. *Tang. half their Sum* :
Tang. half their Difference.

For the mean Arch = $\frac{1}{2}$ Sum of the Extreams.

Cor. 2. In any two Arches (A F, A P,)
The Sum of their Coſines :
Their Diff. (Coſ. leſſer — Coſ. greater) : :
As Cotang. half their Sum :
Tang. half their Difference.

For C H+C G or 2 C B : C H—C G or 2 D F : : C B : B H : : R O : O F : : S N : N K.

Cor. 3. In any two Arches (A F, A P,) as

S. greater : *S. leſſer* : : *T.* $\frac{1}{2}$ *Sum* + *T.* $\frac{1}{2}$ *Diff.* : *T.* $\frac{1}{2}$ *Sum* —*T.* $\frac{1}{2}$ *Diff.*

For by the Demonſtration of this Prop. PG+FH : P G—F H : : N M : N K. And by Compoſition and Diviſion P G : F H : : Q M : K M.

Cor. 4. In any two Arches (A N, N P,) as
The Sum of their Tangents :
Their Difference, (Tan. greater — Tan. leſſer) : :
Sine of their Sum :
Sine of their Difference.

Let A N be the greater, and make N F = N P, then by Similar Triangles, QM : KM : : (PL : FL ::) P G : F H.

Cor. 5. In any two Arches (A N, N P,) as

Tan. greater : *Tan. leſſer* : : *S. Sum* + *S. Difference* : *S. Sum* — *S. Difference.*

For let A N be the greater, and make N F = N P, then A P is the Sum, and, A F is the Difference of the Arches; and by Cor. 4. we have Q M : K M : : P G : F H. And by Compoſition and Diviſion, $\frac{QM+KM}{2}$ or N M : $\frac{QM-KM}{2}$ or N Q : : P G+F H : P G—F H.

Cor. 6. Let A be any Arch leſs than 45°, then
S.$\overline{45+A}$+ *Coſ.* $\overline{45+A}$:
S.$\overline{45+A}$— *Coſ.* $\overline{45+A}$: :

Radius :
Tan. A.

This appears by this Prop. putting $AN=45^\circ$, and $NF=NP=A$.

SCHOLIUM.

The Propoſitions here delivered hold univerſally, obſerving to take negative Coſines and Tangents for Arches greater than a Quadrant.

PROP. VIII.

In any two Arches, AE, ED;
Radius Square — Rectangle of their Tangents : 5.
Radius Square ::
Sum of their Tangents :
Tangent of the Sum of the Arches.

For by ſimilar Triangles CB or $CA-BA$: CK or $CF-KF$ :: CF : CA. Whence $CA^2-BAC=CF^2-KFC$, and $KFC=CF^2-CA^2+BAC=FA^2+BAC$. Again CB : BK :: CF : FA, and BK : DG :: KF : FD or FA, therefore *ex equo* CB or $CA-AB$: DG :: CFK or FA^2+BAC : AF^2; therefore $\overline{AC-AB}\times AF^2=DG\times FA^2+DG\times BAC$. And $\overline{AB+DG}\times AF^2=AC\times\overline{AF^2-AB\times DG}$. *Q. E. D.*

Cor. 1. Let T, *t* be the Tangents, X, *x*, the Cotangents of two Arches, then $\frac{rr-Tt}{T+t}=\frac{Xx-rr}{X+x}=$ Cotangent of the Sum of the Arches.

For let $r=$ Radius, $\tau=$ Cotangent of the Sum: Then $AC=\frac{rr}{\tau}$, and $\overline{AB+DG}\times rr=\frac{rr}{\tau}\times\overline{rr-AB\times DG}$, whence $\tau=\frac{rr-AB\times DG}{AB+DG}=\frac{\textit{rect. Cot}-rr}{\textit{Sum Cot.}}$

FIG. *Cor.* 2. If one of the Arches be 45°, $t =$ Tang. other Arch, then $\frac{r+t}{r-t} \times r =$ Tang. Sum of the Arches.

SCHOLIUM.

What belongs in general to the Addition of Tangents and Cotangents, is delivered in this Prop. and its Cors. All Caſes where the Arches are greater than a Quadrant will eaſily appear by putting negative Tangents and Cotangents for theſe Arches.

PROP. IX.

5. *In any two Arches* AD, AE, *as*
Radius Square + Rectangle of their Tangents :
Radius Square ::
Difference (or Tangent greater — Tangent leſſer) :
Tangent of the Difference of the Arches.

For you will find as in the laſt Prop. CB : DG :: CFK or $FA^2 + BAC : FA^2$. *Q. E. D.*

Cor. 1. Let T, t be the Tangents, X, x the Cotangents of two Arches; then $\frac{rr+Tt}{T-t} = \frac{rr+Xx}{x-X} =$ Cotangent of the Difference of the Arches.

This appears by putting $\frac{rr}{\tau}$ for the Tangent of the Difference.

Cor. 2. If one of the Arches be 45°, $t =$ Tangent of the other, than $\frac{r-t}{r+t} r =$ Tangent of their Difference = Tan. 45° — Tan. of the other Arch.

SCHOLIUM.

For Arches greater than a Quadrant, put negative Tangents and Cotangents.

PROP. X.

The Secant of an Arch is equal to the Sum of the Tangent of it, and the Tangent of half its Complement.

For let A = Arch, T its Tangent, ſ its Secant. $a = \frac{1}{2}$ Complement, t its Tangent. Then by Schol. Pr. II. $ſ = \frac{rr+tt}{2t}$, and $T = \frac{rr-tt}{2t}$, and $ſ - T = t$, whence Secant of A = Tangent of A + Tang. $\frac{1}{2}$ Complement of A.

Cor. 1. The Secant of an Arch = Cotangent of half the Complement — the Tangent of the Arch.

For $S+T = \frac{rr}{t}$ = Cot. a = Cotang. $\frac{1}{2}$ Compl. of A.

Cor. 2. Half the Sum of the Tangent and Cotangent of an Arch is = Secant of the Difference between that Arch and its Complement.

For by Schol. Pr. II. $\frac{1}{2}$ Tangent + $\frac{1}{2}$ Cotangent = Coſecant of the double Arch, and the Complement of the double Arch is the ſame as the Difference between the Arch and its Complement.

Cor. 3. As Radius + Secant : Radius :: Tangent : Tangent of half the Arch. For $r + \frac{\tau+t}{\tau-t} r$ or $\frac{2r\tau}{\tau-t} : r :: \frac{2rr}{\tau-t} : t$. See Sch. Pr. II.

Cor. 3. Secant of an Arch A = Tangent of $\overline{45 + \frac{1}{2}A}^{\circ}$ — Tangent of A. This follows from Cor. 1.

SCHOLIUM.

From what has been before laid down, it will not be difficult to find the Sines of as many Arches as we will, expreſs'd in ſurd Numbers. As in theſe following, where the Prop. in the Margin ſhows whence and how they are derived, either immediately, or by help of what Step.

FIG.		
Pr. IV, 7. Step 8.	1	S. $15^\circ = \frac{1}{2}r\sqrt{2-\sqrt{3}}$.
Pr. IV, 7. and II. 3.	2	S. $18 = r \times \frac{\sqrt{5}-1}{4}$.
Pr. IV, 7; Step 6.	3	S. $22\frac{1}{2} = \frac{1}{2}r\sqrt{2-\sqrt{2}}$.
Pr. III, Step 12.	4	S. $30 = \frac{1}{2}r$.
Pr. IV, 7. Step 2.	5	S. $36 = \frac{1}{4}r\sqrt{10-2\sqrt{5}}$,
Pr. IV, 7.	6	S. $45 = \frac{1}{2}r\sqrt{2}$.
Pr. IV, 7. Step. 2.	7	S. $54 = r \times \frac{\sqrt{5}+1}{4}$.
Pr. IV, 7. Step 4.	8	S. $60 = \frac{1}{2}r\sqrt{3}$.
Pr. IV, 7. Step 6.	9	S. $67\frac{1}{2} = \frac{1}{2}r\sqrt{2+\sqrt{2}}$.
Pr. I, 1. Step 2.	10	S. $72 = \frac{1}{4}r\sqrt{10+2\sqrt{5}}$.
Pr. IV, 7. Step 8.	11	S. $75 = \frac{1}{2}r\sqrt{2+\sqrt{3}}$.
Def. 2.	12	S. $90 = r$.

And thus you may find as many Sines as you pleaſe by Prop. III. Cor. 6, 7, 8, 9. And Prop. IV. Cor. 7. and Prop. V. Cor. 4, 5. by the help of theſe already found; but then they will be ſtill more and more compounded with ſurd Quantities, except you chuſe to extract them. The Sine of 18 in Step the ſecond is not ſo eaſily derived as the Reſt, it is had thus. Put $s=$S.18. Then by Pr. IV. 7. $S.36=\sqrt{\frac{rr-rs}{2}}$, and by Prop. II. 3, the ſame Sine of $36=\frac{2s}{r}\sqrt{rr-ss}$; which equated and divided by $\sqrt{r-s}$, and then reduced, you have a Cubic Equation $8s^3+8rss=r^3$ And the Root is $s=\frac{\sqrt{5}-1}{4}\times r$.

SECT. II.

The Calculation of natural Sines, Tangents, and Secants of Arches.

PROP. XI.

The Radius and Sine AB *of an Arch* DB *being given; to find the Arch.* FIG. 6.

Take the Arch B*b* infinitely ſmall, and draw B*s* ∥ CD, and *bsu* ∥ BA; and let CB or CD $= r$, AB $= y$, DB $= z$, CA $= \sqrt{rr - yy}$, B*b* $= \dot{z}$, *sb* $= \dot{y}$. The Triangles CAB, and B*sb* are ſimilar: Therefore CA $(\sqrt{rr-yy})$: CB (r) :: *sb* $(\dot{y})$: B*b* or $\dot{z} = \frac{r\dot{y}}{\sqrt{rr-yy}}$. And the Fluent is $z = y + \frac{y^3}{3 \cdot 2r^2} + \frac{3 \cdot y^5}{5 \cdot 2 \cdot 4r^4} + \frac{3 \cdot 5y^7}{7 \cdot 2 \cdot 4 \cdot 6r^6} + \frac{3 \cdot 5 \cdot 7y^9}{9 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot r^8}$ &c. or Arch DB $= y + \frac{yy}{2 \cdot 3rr}A + \frac{3 \cdot 3y^2}{4 \cdot 5rr}B + \frac{5 \cdot 5y^2}{6 \cdot 7r^2}C + \frac{7 \cdot 7y^2}{8 \cdot 9r^2}D$ &c. putting A, B, C, &c. for the firſt, ſecond, third Terms, &c.

Cor. 1. If $d =$ Diameter, $c =$ Cord of an Arch, then the Arch $= c + \frac{cc}{2 \cdot 3dd}A + \frac{3^2c^2}{4 \cdot 5d^2}B + \frac{5^2c^2}{6 \cdot 7d^2}C$ &c.

Cor. 2. Let Q $=$ a Quadrant, then the Arch whoſe Coſine is y is $= Q - y - \frac{y^3}{2 \cdot 3r^2} - \frac{3y^5}{2 \cdot 4 \cdot 5r^4} - \frac{3 \cdot 5y^7}{2 \cdot 4 \cdot 6 \cdot 7r^6}$ &c.

Or the Arch whoſe Coſine is y is $= \frac{r-y}{1} + \frac{r^3-y^3}{2 \cdot 3r^2} + 3 \times \frac{r^5-y^5}{2 \cdot 4 \cdot 5r^4} + 3 \times 5 \times \frac{r^7-y^7}{2 \cdot 4 \cdot 6 \cdot 7r^6}$, &c.

PROP.

FIG. 6.

PROP. XII.

The Arch BD *being given; to find the Sine* BA.

Let $CD=r$, $AB=y$, $DB=a$. Then by Prop. XI. $a=y+\frac{y^3}{6r^2}+\frac{3y^5}{40r^4}+\frac{5y^7}{112r^6}+$ *&c.* By the Method of Reverſion of Series, ſuppoſe $y=ba+ca^3+da^5+ea^7$, *&c.* Then

$$\left.\begin{array}{llllll} y = & ba + & ca^3 + & da^5 & + \quad ea^7 & \&c. \\ +\frac{y^3}{6rr} = & & \frac{b^3a^3}{6rr} + & \frac{cb^2}{2rr}a^5 + & \frac{b^2d+ccb}{2rr}a^7 & \&c. \\ +\frac{3y^5}{40r^4} = & & & +\frac{3b^5}{40r^4}a^5 + & \frac{3cb^4}{8r^4}a^7 & \&c. \\ +\frac{5y^7}{112r^6} = & & & & +\frac{5b^7}{112r^6}a^7, & \&c. \\ \&c. & & & & & \end{array}\right\} = a.$$

then equating the homologous Powers of a, $ba=a$, and $b=1$. Alſo $c+\frac{1}{6rr}=0$, and $c=-\frac{1}{6rr}=\frac{-1}{2.3r^2}$. Alſo $d+\frac{c}{2rr}+\frac{3}{40r^4}=0$. Whence $d=\frac{1}{120r^4}=\frac{1}{2.3.4.5r^4}$. Likewiſe $e=\frac{-1}{5040r^6}=\frac{-1}{2.3.4.5.6.7r^6}$, *&c.* Whence y or $AB=a-\frac{a^3}{2.3rr}+\frac{a^5}{2.3.4.5r^4}-\frac{a^7}{2.3.4.5.6.7r^6}+$ *&c.*

Or Sine $AB=a-\frac{aa}{2.3r^2}A-\frac{aa}{4.5r^2}B-\frac{aa}{6.7r^2}C-\frac{aa}{8.9r^2}D$, *&c.* where A, B, C, *&c.* are the foregoing Terms with their Signs.

Cor. 1. If a be any Arch, its Coſine $CA=r-\frac{aa}{1.2r^2}A-\frac{aa}{3.4r^2}B-\frac{aa}{5.6r^2}C-\frac{aa}{7.8r^2}D-$ *&c.*

For $CA = \sqrt{rr - yy} = \sqrt{rr - a^2 + \frac{a^4}{3r^2} - \frac{2a^6}{45r^4} + \frac{a^8}{315r^6} - \&c.} = r - \frac{aa}{1.2r} + \frac{a^4}{1.2.3.4r^3} - \frac{a^6}{1.2.3.4.5.6r^5} + \frac{a^8}{1.2\ldots.8r^7} - \&c.$

Cor. 2. If d be the Diameter, a any Arch; then the Cord $= a - \frac{aa}{2.3d^2}A - \frac{a^2}{4.5d^2}B - \frac{a^2}{6.7d^2}C - \frac{a^2}{8.9d^2}D$, &c.

PROP. XIII.

The Sine and Coſine of an Arch A *being given, and if there be given another Arch* z; *to find the Sine and Coſine of the Sum of the Arches* A+z, *and alſo of the Difference* A—z.

Let s, c be the Sine and Coſine of A. And by Prop. XII. and Cor. 1. the Sine of the Arch z is $z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4}$, &c. $=$ B. And its Coſine $r - \frac{z^2}{2r} + \frac{z^4}{24r^3}$ &c. $=$ D. Then by Prop. V. and Cor. 1. and Prop. VI. and Cor. $\frac{sD + cB}{r} = S.\overline{A+z}$. And $\frac{cD - sB}{r} =$ Coſ. $\overline{A+z}$. And $\frac{sD - cB}{r} = S.\overline{A-z}$. And $\frac{sB + cD}{r}$ $=$ Coſ. A—z, that is,

$$S.\overline{A+z} = s + \frac{cz}{r} - \frac{sz^2}{1.2r^2} - \frac{cz^3}{2.3r^3} + \frac{sz^4}{1.2.3.4r^4} + \frac{cz^5}{1.2.3.4.5r^5} - \&c.$$

$$\text{Coſ. } \overline{A+z} = c - \frac{sz}{r} - \frac{cz^2}{2r^2} + \frac{sz^3}{2.3r^3} + \frac{cz^4}{2.3.4r^4} - \frac{sz^5}{2.3.4.5r^5} - \&c.:$$

S.A—z

$$S.\overline{A-z}=s-\frac{cz}{r}-\frac{sz^2}{2r^2}+\frac{cz^3}{2.3r^3}+\frac{sz^4}{2.3.4r^4}-\frac{cz^5}{2.3.4.5r^5}$$
— *&c.*

$$Coſ.\ \overline{A-z}=c+\frac{sz}{r}-\frac{cz^2}{2r^2}-\frac{sz^3}{2.3r^3}+\frac{cz^4}{2.3.4r^4}+\frac{sz^5}{2.3.4.5r^5}$$
— *&c.*

Cor. 1. If there be given s, c, the Sine and Coſine of an Arch A; and $s+x$ be the Sine of another Arch $\overline{A+z}$; then the Difference of the Sines $x=\frac{cz}{r}-\frac{sz^2}{2r^2}-\frac{cz^3}{2.3r^3}+\frac{sz^4}{2.3.4r^4}$, *&c.* And the Difference of the Arches $z=\frac{r}{c}\times : x+\frac{sx^2}{2c^2}+\frac{rr+2ss}{6c^4}x^3+\frac{3r^2+2ss}{8c^6}sx^4$ *&c.*

For by this Prop. the Sine of $\overline{A+z}$, or $s+x=s+\frac{cz}{r}-\frac{sz^2}{2rr}$, *&c.* and $x=\frac{cz}{r}-\frac{sz^2}{2rr}$, *&c.* and by Reverſion of Series z is found as above.

Cor. 2. The ſecond Difference of the Sines of $A-z$, A, $A+z$, is $=2s\times:\frac{z^2}{1.2r^2}-\frac{z^4}{1.2.3.4r^4}+\frac{z^6}{1.2.3.4.5.6r^6}$ — *&c.* (for in any three Things, the ſecond Difference is equal to the Difference between twice the mean, and the Sum of the Extreems.)

PROP. XIV.

6. *The verſed Sine of an Arch* DB *being given; to find the Arch.*

Let $CD=r$. $DA=v$. $Au=\dot{v}$. $DB=z$. Then $AB=\sqrt{2rv-vv}$. The Triangles CAB and Bsb are ſimilar, therefore $\sqrt{2rv-vv}:r::\dot{v}:Bb$ or $\dot{z}=\frac{r\dot{v}}{\sqrt{2rv-vv}}$. And the Fluent is $z=\sqrt{2rv}\times:1+$

$\frac{v}{3.2^2 r}+\frac{2v^2}{2^3.4.5r^2}+\frac{3.5v^3}{2^4.4.6.7r^3}$, *&c.* that is Arch DB $=\sqrt{2rv}+\frac{v}{3.4r}A+\frac{3^2v}{5.8r}B+\frac{5^2v}{7.12r}C+\frac{7^2v}{9.16r}D$, *&c.*

Or rather DB$=\sqrt{dv}+\frac{v}{2.3d}A+\frac{3^2v}{4.5d}B+\frac{5^2v}{6.7d}C$ *&c.* putting $d=$ Diameter. Where A, B, C are the foregoing Terms.

Cor. 1. In any Arch a, the verſed Sine is $=\frac{aa}{2r}-\frac{aa}{3.4r^2}A-\frac{aa}{5.6r^2}B-\frac{aa}{7.8r^2}C-$ *&c.* where A, B, C are the foregoing Terms with their Signs.

For Radius — Coſine is equal to the verſed Sine, and by Cor. 1. Pr. XII. the Coſine is $r-\frac{a^2}{1{:}2r}+\frac{a^4}{1.2.3.4r^3}$, *&c.*

Cor. 2. The coverſed Sine of the Arch a, is $=r-a+\frac{a^3}{2.3r^2}-\frac{a^5}{2.3.4.5r^4}+\frac{a^7}{2.3.4.5.6.7r^6}$, *&c.* for it is $=$ Radius — Sine.

Cor. 3. Verſed Sine of the Supplement is $=2r-\frac{a^2}{1.2r^2}A-\frac{aa}{3.4r^2}B-\frac{aa}{5.6rr}C-\frac{aa}{7.8r^2}D$, *&c.* for it is = Radius + Coſine.

PROP. XV.

The Tangent DT *of an Arch* DB *being given; to find the Arch.* 6.

For draw Ct infinitely near CT, and T$n\perp$Ct, and let CD$=r$, DT$=t$, T$t=\dot{t}$, DB$=z$, CT$=\sqrt{rr+tt}$.

The Triangles CDT and Ttn are ſimilar, as alſo CBb, and CTn, whence CT $: r :: \dot{t} :: n$T$=\frac{r\dot{t}}{\text{CT}}$, and

FIG. 6.

and CT : nT or $\frac{r\dot{t}}{CT}$ :: r : B$b = \frac{rr\dot{t}}{CT^2}$, that is $\dot{z} = \frac{rr\dot{t}}{rr+tt}$.

And the Fluent is $z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} +$, &c. that is, Arch DB $= t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8}$, &c.

Cor. 1. *In any Arch a the Tangent is* $= a + \frac{a^3}{3rr} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} + \frac{1382a^{11}}{155925r^{10}} + \frac{21844a^{13}}{6081075r^{12}} + \frac{929569a^{15}}{638512875r^{14}} +$ &c.

This will appear by Reverſion of the Series $a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4}$, &c. Or rather by multiplying the Sine by the Radius, and dividing by the Coſine, thus

$$\frac{ra - \frac{a^3}{6r} + \frac{a^5}{120r^3}, \text{ \&c.}}{r - \frac{aa}{2r} + \frac{a^4}{24r^3}, \text{ \&c.}} = \text{Tangent.}$$

Cor. 2. *The Cotangent of an Arch a is* $= \frac{rr}{a} - \frac{a}{3} - \frac{a^3}{45r^2} - \frac{2a^5}{945r^4} - \frac{a^7}{4725r^6} - \frac{2a^9}{93555r^8} - \frac{1382a^{11}}{638512875r^{10}} - \frac{4a^{13}}{18243225r^{12}} -$ &c. as will appear by dividing Radius Square by the Value of the Tangent in Cor. 1. Or the Coſine by the Sine, and then multiplying by Radius.

Cor. 3. Hence alſo if τ be the Cotangent of an Arch, then the Arch $= \frac{rr}{\tau} - \frac{r^4}{3\tau^3} + \frac{r^6}{5\tau^5} - \frac{r^8}{7\tau^7} + \frac{r^{10}}{9\tau^9} -$ &c. For $t = \frac{rr}{\tau}$.

PROP.

FIG.

PROP. XVI.

The Secant CT *of an Arch* DB *being given; to find the Arch.* 6.

Let $CD=r$, $CT=ſ$, $nt=\dot{ſ}$, $DT=\sqrt{ſſ-rr}$, $DB=z$, then by ſimilar Triangles $DT:r::\dot{ſ}:nT=\frac{r\dot{ſ}}{DT}$, and $ſ:r::nT$ or $\frac{r\dot{ſ}}{DT}:\dot{z}=\frac{rr\dot{ſ}}{ſ\times DT}$ or $\dot{z}=\frac{rr\dot{ſ}}{ſ\sqrt{ſſ-rr}}$, and the Fluent is $z=-\frac{rr}{ſ}-\frac{r^4}{2.3ſ^3}-\frac{3r^6}{2.4.5ſ^5}-\frac{3.5r^8}{2.4.6.7ſ^7}$, &c. but in D, $z=0$, and $ſ=r$, therefore the Fluent corrected is, Arch DB or $z=r\times:\frac{ſ-r}{ſ}+\frac{ſ^3-r^3}{2.3ſ^3}+\frac{3.\overline{ſ^5-r^5}}{2.4.5ſ^5}+\frac{3.5.\overline{ſ^7-r^7}}{2.4.6.7ſ^7}+$ &c.

Or thus, in the Point F, $z=$ a Quadrant Q, and $ſ=$ infinity. Therefore the whole Series $=0$. Therefore by Correction $z-Q=0-\frac{rr}{ſ}-\frac{r^4}{2.3ſ^3}$, &c. whence Arch z or $DB=Q-\frac{rr}{ſ}-\frac{r^4}{2.3ſ^3}-\frac{3r^6}{2.4.5ſ^5}-\frac{3.5r^8}{2.4.6.7ſ^7}$ — &c.

Cor. 1. If σ be the Coſecant of an Arch, then the Arch is $=\frac{rr}{\sigma}+\frac{r^4}{2.3\sigma^3}+\frac{3r^6}{2.4.5\sigma^5}+\frac{3.5r^8}{2.4.6.7\sigma^7}+$ &c. for this is the Complement of the Arch whoſe Secant is σ, by this Prop.

Cor. 2. In any Arch a, the Secant $=r+\frac{aa}{2r}+\frac{5a^4}{24r^3}+\frac{61a^6}{720r^5}+\frac{277a^8}{8064r^7}+\frac{50521a^{10}}{3628800r^9}+\frac{540553a^{12}}{95800320r^{11}}+$ &c.

For Radius Square (rr), divided by the Coſine $(r-\frac{aa}{2}+\frac{a^4}{24r^3}$, &c.) is = Secant.

Cor. 3. The Coſecant of an Arch a is $= \frac{rr}{a} + \frac{a}{6} + \frac{7a^3}{360r^2} + \frac{31a^5}{15120r^4} + \frac{127a^7}{604800r^6} + \frac{73a^9}{3421440r^8} + \frac{1414477a^{11}}{653837184000r^{10}} +$ &c. for Radius Square divided by the Sine = Coſecant.

SCHOLIUM.

If an Arch be given in Degrees, its length may be found thus: When the Radius of a Circle is 1, half the Circumference is 3.14159265358979, therefore $\frac{3.14159 \text{ \&c.}}{180} =, 01745329251994 =$ length of 1 Degree. Therefore if r be the Radius of any Circle, then $r \times, 01745$ &c. = length of the Arch of 1 Degree in that Circle. Conſequently $r \times, 01745329$ &c. × Number of Degrees and decimal Parts, gives the length of the Arch. And this muſt be taken for, a or z in the foregoing Propoſitions.

SECT. III.

The Calculation of logarithmic Sines, Tangents, and Secants.

Tho' the finding the logarithmic Sines, Tangents, *&c.* is no more than finding the Logarithms of the natural ones by a Table of Logarithms: Yet theſe logarithmic Sines, Tangents, *&c.* may be found without the Table of Logarithms, or elſe without the natural Sines or Tangents, and ſometimes without either, by having only the Arch given.

The logarithmic or artificial Sines, Tangents and Secants are calculated to the Radius 1 with 10 Cyphers annext, *viz.* 10000000000; ſo that the Log. Radius will therefore be 10. But in calculating theſe Sines, Tangents, or Secants, we can more eaſily compute them for the Radius 1, and then adding 10 gives the Log. Sine, Tangent, or Secant, to the Radius of the Tables. Therefore

In a Circle whoſe Radius is 1, the Log. of that Radius is 0, and the Length of an Arch of 1 Degree $=\frac{3.1415926536}{180}=,0174532925$2, and this Number multiply'd into any Number of Degrees gives the Length of the Arch of theſe Degrees, for the Radius 1. and this Length muſt be uſed in the following Propoſitions, when the Degrees are given.

Now if you put M=,4342944819, and having any Quantity given (expreſſing a Sine, Tangent, *&c.*) you have no more to do but to divide the Fluxion of that Quantity by the Quantity itſelf, and find the Fluent by infinite Series, which multiply'd by M is the Logarithm of the Quantity required.

PROP. XVII.

To find the logarithmic or artificial Sine of the Arch x.

Let $y=$ nat. Sine of x, then by Prop. XII. $y=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}$, &c. where $r=1$. And $\dot{y}=\dot{x}-\frac{x^2\dot{x}}{2}+\frac{x^4\dot{x}}{24}-\frac{x^6\dot{x}}{720}$, &c. then by Diviſion $\frac{\dot{y}}{y}=\frac{\dot{x}}{x}-\frac{x\dot{x}}{3}-\frac{x^3\dot{x}}{45}-\frac{2x^5\dot{x}}{945}-\frac{x^7\dot{x}}{4725}$, &c. and Fluent of $M\frac{\dot{y}}{y}$, or the Log. y $=$ Log. $x-M\times:\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2835}+\frac{x^8}{37800}+$ &c. to which add 10, and you have the logarithmic Sine in the Tables; and that without knowing the natural Sine.

Otherwiſe.

Let $z=\frac{1-y}{1+y}$, where $y=$ nat. Sine; then will $y=\frac{1-z}{1+z}$, and $\frac{\dot{y}}{y}=\frac{-2\dot{z}}{1-zz}=-2\times:\dot{z}+z^2\dot{z}+z^4\dot{z}+z^6\dot{z}$ &c. whence Fl. $M\frac{\dot{y}}{y}$, or Log. $y=-2M\times z+\frac{1}{3}z^3+\frac{1}{5}z^5+\frac{1}{7}z^7$ &c. And the log. Sine of the Tables $=10-2M\times:z+\frac{1}{3}z^3+\frac{1}{5}z^5+\frac{1}{7}z^7$ &c. which is had without the Table of Logarithms.

Or thus.

Let $y=$ nat. Sine, $z=$ Coſine. Then $y=\sqrt{1-zz}$, and $\frac{\dot{y}}{y}=\frac{-z\dot{z}}{1-zz}=-z\dot{z}-z^3\dot{z}-z^5\dot{z}$, &c. whence Fl. $\frac{M\dot{y}}{y}=-\frac{1}{2}M\times:z^2+\frac{z^4}{2}+\frac{z^6}{3}+\frac{z^8}{4}$, &c. and log. Sine of the Tables $=10-\frac{1}{2}M\times:z^2+\frac{1}{2}z^4+\frac{1}{3}z^6+\frac{1}{4}z^8$, &c.

And thus if you have any other Line or Quantity in the Circle, given, by which you can compute the

Sine, you may, from that, find the log. Sine. And the like may be done for the Tangent, or Secant.

Cor. Hence if $s=$ log. Sine of $\frac{1}{2}$A, half an Arch given; then the Log. verſed Sine of A$=.3010300+2s-10$, for the Tables. For Rad. × verſed Sine $=$ 2 Sine Square of the half Arch, by Cor. 2. Prop. II. and $.3010300=$ Log. 2.

Or thus. Log. verſ. of Arch $x=10.3010300+2$ log. $\frac{1}{2}x-2\mathrm{M}\times:\frac{x^2}{4.6}+\frac{x^4}{4^2.180}+\frac{x^6}{4^3.2835}+\frac{x^8}{4^4.37800}$ &c. for s (or the log. Sine of $\frac{1}{2}x$) $=10+$ log. $\frac{1}{2}x$, $-\mathrm{M}\times\frac{x^2}{4.6}+\frac{x^4}{16.180}$, &c.

Or laſtly, If $z=$ Coſine, then log. verſ. $=10-\mathrm{M}\times:z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}$, &c. For $1-z=$ verſed Sine, and its Fluxion $\frac{-\dot{z}}{1-z}=-\dot{z}-z\dot{z}-z^2\dot{z}$, &c. whoſe Fluent is $-\mathrm{M}\times:z+\frac{1}{2}z^2+\frac{1}{3}z^3$, &c. to which add 10.

P R O P. XVIII.

The Arch x being given to find the log. Coſine.

By Cor. 1. Pr. XII. the Coſine $y=1-\frac{xx}{2}+\frac{x^4}{24}-\frac{x^6}{720}$ &c. therefore $\frac{\dot{y}}{y}=-x\dot{x}-\frac{1}{3}x^3\dot{x}-\frac{2}{15}x^5\dot{x}-\frac{17}{315}x^7\dot{x}$, &c. and the Fl. $\frac{\mathrm{M}\dot{y}}{y}$ or Log. $y=-\mathrm{M}\times:\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\frac{17}{2520}x^8$, &c. And the tab. Coſ. of the Arch x is $=10-\mathrm{M}\times:\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\frac{17}{2520}x^8+$ &c. And that without either the nat. Coſine or the Table of Logarithms.

Or the log. Coſine may be found in the ſame Manner as the Sine, in the two laſt Methods in Pr. XVII.

FIG.

PROP. XIX.

Given an Arch z, and the Sine and Coſine of an Arch A; *to find the log. Sine and Coſine of the Sum* $\overline{A+z}$, *and alſo of the Difference* $\overline{A-z}$.

Let the nat. Sine and Coſine of A be s, c. $y = S.\overline{A+z}$. Then by Prop. XIII. $y = s + \frac{cz}{1} - \frac{sz^2}{2} - \frac{cz^3}{6} + \frac{sz^4}{24}$ &c. and $\frac{\dot{y}}{y} = \frac{c\dot{z}}{s} - \frac{z\dot{z}}{ss} + \frac{cz^2\dot{z}}{s^3} - \frac{1+2cc}{3s^4}z^3\dot{z}$, &c. and the Fl. $M\frac{\dot{y}}{y}$ or Log. $y = M\times : \frac{cz}{s} - \frac{z^2}{2ss} + \frac{cz^3}{3s^3} - \frac{1+2cc}{12s^4}z^4$, &c. but when $z=0$, $y=s$, therefore by Correction,

Log. y — Log. $s = M\times : \frac{cz}{s} - \frac{z^2}{2ss}$ &c. whence the

Log. $S.\overline{A+z} =$ *Log.*$s + M\times : \frac{cz}{s} - \frac{z^2}{2ss} + \frac{cz^3}{3s^3} - \frac{1+2cc}{12s^4}z^4$, &c. and after the the ſame manner

Log. $S.\overline{A-z} =$ *Log.* $s - M\times : \frac{cz}{s} + \frac{z^2}{2ss} + \frac{cz^3}{3s^3} + \frac{1+2cc}{12s^4}z^4$, &c.

Log. Coſ. $\overline{A+z} =$ *Log.* $c - M\times : \frac{sz}{c} + \frac{z^2}{2cc} + \frac{sz^3}{3c^3} + \frac{1+2ss}{12cc}z^4$, &c.

Log. Coſ. $\overline{A-z} =$ *Log.* $c + M\times : \frac{sz}{c} - \frac{z^2}{2cc} + \frac{sz^3}{3c^3} - \frac{1+2ss}{12c^4}z^4$, &c.

PROP.

PROP. XX.

The Arch x being given to find its Log. Tangent.

By Cor. 1. Pr. XV. the Tangent $t=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7$, *&c.* therefore $\frac{\dot{t}}{t}=\frac{\dot{x}}{x}+\frac{2}{3}x\dot{x}+\frac{14}{45}x^3\dot{x}+\frac{124}{945}x^5\dot{x}$, *&c.* whence Log. Tan. $t=$ Log. $x+\mathrm{M}\times:\frac{1}{3}x^2+\frac{7}{90}x^4+\frac{62}{2835}x^6+\frac{127}{18900}x^8$, *&c.* to which add 10.

Otherwiſe.

To find the Log. Tan. of the Arch A$+x$, *the Tan. of the Arch* A *being given.*

Let Tan. A$=a$, Tan. $x=t$, $y=$ Tan. $\overline{A+x}$. Then by Prop. VIII. $y=\frac{a+t}{1-at}$, and $\dot{y}=\frac{1-aa}{\overline{1-at}^2}\dot{t}$, and $\frac{\dot{y}}{y}=\frac{\overline{1+aa}\,.\,\dot{t}}{a+\overline{1-aa}\,.\,t-att}$. Let $b=\frac{1+aa}{a}$, $c=\frac{1-aa}{a}$, then $\frac{\dot{y}}{y}=b\times:\dot{t}-ct\dot{t}+\overline{1+cc}\,.\,tt\dot{t}-\overline{2c+c^3}\,.\,t^3\dot{t}$, *&c.* and Fl. $\mathrm{M}\frac{\dot{y}}{y}$ or Log. $y=b\mathrm{M}\times:t-\frac{1}{2}ctt+\frac{1+cc}{3}t^3-\frac{2c+c^3}{4}t^4$, *&c.* but when x and $t=0$, $y=a$, then by Correction Log. $y-$ Log. $a=b\mathrm{M}\times:t-\frac{1}{2}ctt$, *&c.* that is

Log. Tan. $\overline{A+x}=$ *Log. Tan.* A$+b\mathrm{M}\times:t-\frac{1}{2}ctt+\frac{1+cc}{3}t^3-\frac{2c+c^3}{4}t^4$, *&c.* and the ſame for the *Log. Tan.* $\overline{A-x}$, changing the Sines of the odd Powers of t.

But if A$=45°$, then $a=1$, and $\frac{\dot{y}}{y}=\frac{2\dot{t}}{1-tt}=2\dot{t}+2t^2\dot{z}+2t^4\dot{z}$, *&c.* And conſequently Log. y or Fl. $\frac{\mathrm{M}\dot{y}}{y}=2\mathrm{M}\times:t+\frac{1}{3}t^3+\frac{1}{5}t^5$, *&c.* that is

$$\textit{Log. Tan. } \overline{45+x}=2\text{M}\times:t+\frac{t^3}{3}+\frac{t^5}{5}+\frac{t^7}{7},\ \&c.$$

But if inſtead of t you put its Value $x+\frac{1}{3}x^3+\frac{2}{15}x^5$, &c. and the Fluxion for the Fluxion, then will $\frac{2\dot{t}}{1-tt}$ $=2\times:\dot{x}+2x^2\dot{x}+\frac{10}{3}x^4\dot{x}+\frac{244}{45}x^6\dot{x}$ &c. whence Log. y, or *Log. Tan.* $\overline{45+x}=2\text{M}\times:x+\frac{2}{3}x^3+\frac{2}{3}x^5+\frac{244}{315}x^7$ &c. and likewiſe

$$\textit{Log. Tan. } \overline{45-x}=-2\text{M}\times:t+\frac{t^3}{3}+\frac{t^5}{5}\ \&c.=-2\text{M}\times:x+\frac{2}{3}x^3+\frac{2}{3}x^5,\ \&c.$$

to each of which add 10 for the tabular Log. Tang.

Cor. The Log. Cot. of $x=10-\text{Log. }x-\text{M}\times:\frac{x^2}{3}+\frac{7}{90}x^4+\frac{62}{2835}x^6+\frac{127}{18900}x^8$ &c.

And Log. Cot. $\overline{\text{A}+x}=$ is the ſame as the *Log. Tan.* of $\overline{\text{A}-x}$, found before.

PROP. XXI.

To find the logarithmic Secant of a given Arch.

Let x be the Arch, y its Secant; then by Cor. 2. Prop. XVI. $y=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6$, &c. whence $\frac{\dot{y}}{y}=x\dot{x}+\frac{1}{3}x^3\dot{x}+\frac{2}{15}x^5\dot{x}+\frac{17}{315}x^7\dot{x}$, &c. then Log. y or Fluent $\text{M}\frac{\dot{y}}{y}=\text{M}\times:\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\frac{17x^8}{2520}+\frac{31}{14175}x^{10}$ &c. *hence the tab. log. Secant*$=10+\text{M}\times:\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\frac{17x^8}{2520}$, &c.

Or thus.

To find the Log. Secant of $\overline{45+z}$. let Log. Secant $45=s$, by Prop. XIX. log. Coſ. $\overline{45+z}=$ log. Coſ. $45-M\times:\frac{sz}{c}+\frac{z^2}{2cc}$, *&c.* = (becauſe $s=c$, $r=1$.) log. Coſ. $45-M\times:z+zz$, *&c.* but by Cor. 2. Pr. I. Secant × Coſine $=1$. Or log. Secant $=-$ log. Coſine. Therefore, log. Sec. $\overline{45+z}=$ log. Sec. $45+M\times:sz+z^2+$, *&c.* that is

Log. Sec. $\overline{45+z}=s+M\times:z+z^2+\frac{2}{3}z^3+\frac{1}{3}z^4$, *&c.*

Cor. The log. Coſecant of an Arch x *is* $=10-$ Log. $x+M\times:\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2835}+\frac{x^8}{37800}$, *&c.* For log. Coſecant $=-$ log. Sine, when Radius is 1.

Or log. Coſecant of $\overline{45+z}$, *or log. Secant of* $\overline{45-z}=s-M\times:z-z^2+\frac{2}{3}z^3-\frac{1}{3}z^4$, *&c.* by this Prop. changing the Signs of the odd Powers of z.

SCHOLIUM I.

Having the log. Sine, Tangent or Secant giver, the Arch belonging thereto may be found by the Reverſion of Series. As ſuppoſe there is given T the *log. Tan.* $\overline{45+x}$, then $T=10+2M\times:x+\frac{2}{3}x^3+\frac{2}{5}x^5$ *&c.* by Pr. XX, and $x+\frac{2}{3}x^3+\frac{2}{5}x^5$, *&c.* $=\frac{T-10}{2M}=l$ ſuppoſe, then by Reverſion $x=l-\frac{2}{3}l^3+\frac{14}{15}l^5$ *&c.* and

$$\frac{x}{,0174532}=\text{Degrees in } x.$$

Again, let there be given K, the log. Secant of $\overline{45+z}$, then by Prop. XXI. $K=s+M\times:z+z^2+\frac{2}{3}z^3$, *&c.* then $z+z^2+\frac{2}{3}z^3$, *&c.* $=\frac{K-s}{M}=l$. Then by Reverſion $z=l-ll+\frac{4}{3}l^3$, *&c.* which divided by ,0174532 will give the Degrees in z.

SCHOLIUM II.

In any Arch let log. Radius $=r=10$.

$s=$ log. Sine. $t=$ log. Tangent. $ſ=$ log. Secant.
$c=$ log. Coſine. $\tau=$ log. Cotangent. $\sigma=$ log. Coſecant.

Then

$$s=c+t-r=t+r-ſ=c+r-\tau=2r-\sigma.$$
$$c=s+r-t=2r-ſ=s+\tau-r=\tau+r-\sigma.$$
$$t=s+r-c=2r-\tau.$$
$$\tau=c+r-s=2r-t.$$
$$ſ=t+r-s=3r-s-\tau=\sigma+r-\tau=2r-c.$$
$$\sigma=\tau+r-c=3r-c-t=ſ+r-t=2r-s.$$

By what has been delivered in the two laſt Sections, the Sines, Tangents and Secants, of all Arches, whether natural or artificial, may eaſily be found; and from thence the Table of Sines and Tangents may be conſtructed, with great Eaſe and Expedition. Or any particular Numbers in the Tables may be computed anew, and any Errors corrected therein. But the nat. Sines and Tangents being firſt found, the artificial ones are moſt eaſily had by a Table of Logarithms; for this Reaſon I have not continu'd the ſeveral Series in this laſt Section to any great Number of Places; intending rather to give the Reader the Principles of Calculation than the Calculation itſelf; ſince in all Probability there will be few Perſons who will take the Pains to calculate them anew. But if any has a Mind to do it, he may himſelf continue any of theſe Series as far as he pleaſes.

S E C T. IV.

The Calculation of the Sines, Coſines, Cords, &c. of multiple Arches.

P R O P. XXII.

If a Trapezoid BCDE, *whoſe Sides* BE, CD *are parallel, be inſcribed in a Circle*; *and the Cords* BD, CE *be drawn*; *I ſay*, $BD^2-DE^2=BE\times CD$.

FIG. 19.

Make Angle $EBO=CBD$, then $EBD=CBO$; and ſince $BEO=CBD$, therefore the Triangles BCD and BOE are ſimilar, and $BD : CD :: BE : OE$, and $BD\times OE=BE\times CD$. Alſo ſince Angle $BCO=BDE$, the Triangles BCO, and BDE are ſimilar, whence $BD : DE :: BC : CO$, and $BD\times CO=DE\times BC=DE^2$. Therefore $BD\times\overline{OE+OC}=BE\times CD+DE^2$, that is $BD\times CE$ or $BD^2=BE\times CD+DE^2$, or $BE\times CD=BD^2-DE^2=\overline{BD+DE}\times\overline{BD-DE}$.

Cor. 1. If A be any Arch, and BC or $DE=n$ times A, and $BD=m$ times A. Then

As Cord of $\overline{m-n}\,.\,A$:
Cord . $mA+$ Cord . nA ::
Cord . $mA-$ Cord . nA :
Cord . $\overline{m+n}\,.\,A$.

For CD is the Cord of $\overline{m-n}$ times A, and BE of $\overline{m+n}\times A$.

Cor. 2. As Cord of A :
Cord . $\overline{n+1}$.A+ Cord . nA ::
Cord . $\overline{n+1}$.A− Cord . nA :
Cord . $\overline{2n+1}$.A.

This appears by putting $m=n+1$, or $CD=A$.

Cor.

FIG. 19. *Cor.* 3. As Cord of $\overline{m-1}$. A :
Cord . m A $+$ Cord . A : :
Cord . m A $-$ Cord . A :
Cord $\overline{m+1}$. A.

This appears by putting $n=1$, or C B or D E $=$ A.

Cor. 4. If three Arches CD, BCD, BCDE are in arithmetic Progreſſion, $BD^2-BC^2=CD\times BE$.

PROP. XXIII.

22. *In the Circle* AEB, *whoſe Diameter is* AB, *if the Arches* AD, EF, FG *be taken equal to one another, and the Cords* BD, BE, BF, BG *drawn. I ſay,*

As Radius CB : *Cord* BD : : *ſo the middle Cord* BF : BE$+$BG, *the Sum of the extream Cords.*

For produce BE to H, and make FH$=$FB, then $\angle$BHF$=$HBF$=$FBG, and draw the Cords EF, FG. Then the Angle BEF$=$BFG$+$FBG, both ſtanding on the ſame Arch FGB. Alſo BEF$=$EFH$+$EHF$=$EFH$+$FBG, therefore EFH$=$BFG. And the Triangles EFH, FBG are ſimilar and equal. Whence HE$=$GB, and HB$=$EB$+$BG. Alſo the Triangles DCB and HFB are ſimilar, therefore BC : BD : : BF : BH or EB$+$BG.

Cor. If Arch AD$=$A, Arch BF$=n\times$A. Then
As Rad :
Cord . ſup. A : :
Cord . n A :
Cord . $\overline{n-1}$. A $+$ Cord . $\overline{n+1}$. A.

And the ſame holds in reſpect of the Cords of the Supplements of theſe Arches, if AF be ſuppoſed to be made equal to n A.

SCHOLIUM.

If the Cord BG lye on the other Side of B, then BG is negative.

PROP. XXIV.

The Sine and Coſine of an Arch, being given; to find the Sine of n times that Arch.

Let $x=$ Coſine, $y=$ Sine, given. And A the given Arch, and at preſent let Radius $=1$. Then by Cor. 7. Prop. III. $2x\times\text{S}.n\text{A}=\text{S}.\overline{n-1}.\text{A}+\text{S}.\overline{n+1}.\text{A}$, whence $\text{S}.\overline{n+1}.\text{A}=2x\times\text{S}.n\text{A}-\text{S}.\overline{n-1}.\text{A}$. Or becauſe $xx+yy=1$. $\text{S}.\overline{n+1}.\text{A}=2x\times\text{S}.n\text{A}-\overline{xx+yy}\times\text{S}.\overline{n-1}.\text{A}$. That is if any Sine be multiply'd by $2x$, and the next preceding one multiply'd by $xx+yy$, and ſubtracted from it, gives the next following Sine. Thus you will have

$\text{S}.\ \text{A}=y$.
$\text{S}.2\text{A}=2xy$.
$\text{S}.3\text{A}=3x^2y-y^3$.
$\text{S}.4\text{A}=4x^3y-4xy^3$.
$\text{S}.5\text{A}=5x^4y-10x^2y^3+y^5$. *&c.*

In what follows, if a denotes any variable Quantity, then I denote its firſt and ſecond ſucceeding Values by $\underset{1}{a}$, $\underset{2}{a}$, *&c.* and its firſt and ſecond preceding Values by $\underset{-1}{a}$, $\underset{-2}{a}$ *&c.* and ſo of others. And with theſe I proceed according to the Method of Increments in calculating the following Propoſitions.

Suppoſe then any Arch, as nA, is repreſented in general thus, $\text{S}.n\text{A}=nx^{n-1}y-ax^{n-3}y^3+bx^{n-5}y^5-$ *&c.* then according to the Conſtruction of Sines, before mentioned, we ſhall have

$$\left\{\begin{array}{l} 2nx^ny-2ax^{n-2}y^3+2bx^{n-4}y^5-\text{\&c.}=\text{S}.n\text{A}\times 2x, \\ \left.\begin{array}{l} -\underset{-1}{n}x^ny+\underset{-1}{a}x^{n-2}y^3-\underset{-1}{b}x^{n-4}y^5-\text{\&c.} \\ \qquad\qquad -\underset{-1}{n}\qquad\quad +\underset{-1}{a} \end{array}\right\}=\overline{xx+yy}\times\text{S}.\overline{n-1}.\text{A}. \end{array}\right.$$

$$=\overline{n+1}.x^ny-\underset{1}{a}x^{n-2}y^3+\underset{1}{b}x^{n-4}y^5-\text{\&c.}=\text{S}.\overline{n+1}.\text{A}.$$

Hence equating the Coefficients of the homologous Powers, $n+1=2n-\underset{-1}{n}$, and $1=n-\underset{-1}{n}$. That is $n=1$.

likewiſe

FIG. likewiſe $\underset{1}{a}=2a-\underset{-1}{a}+\underset{-1}{n}$, or $\underset{2}{a}=2\underset{1}{a}-a+n$, whence $\underset{2}{a}-2\underset{1}{a}+a=n$, that is $\underset{..}{a}=n$. And the Integral is $\underset{.}{a}=\frac{n\overline{n-1}}{2}$, and the Integral of this is $a=\frac{\overline{n-2}\,n\overline{n-1}}{2\cdot 3}$ (for when $n=3$, $a=\frac{\overline{n-2}\,n\overline{n-1}}{2\cdot 3}=1$). Again $\underset{1}{b}=2b-\underset{-1}{b}+\underset{-1}{a}$, or $\underset{2}{b}=2\underset{1}{b}-b+a$, whence $\underset{2}{b}-2\underset{1}{b}+b=a$, or $\underset{..}{b}=a=\frac{\overline{n-2}\,n\overline{n-1}}{2\cdot 3}$, whence $\underset{.}{b}=\frac{\overline{n-3}\,\overline{n-2}\,n\overline{n-1}}{2\cdot 3\cdot 4}$, and $b=\frac{\overline{n-4}\,\overline{n-3}\,\overline{n-2}\,n\overline{n-1}}{2\cdot 3\cdot 4\cdot 5}$. In like manner $\underset{..}{c}=b$, and $c=\frac{\overline{n-6}\,\overline{n-5}\,\overline{n-4}\,\overline{n-3}\,\overline{n-2}\,n\overline{n-1}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$, *&c. whence*

$$S.nA=nx^{n-1}y-\frac{n\cdot\overline{n-1}\cdot\overline{n-2}}{1\cdot 2\cdot 3}x^{n-3}y^3+\frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}\cdot\overline{n-4}}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5}y^5,$$ *&c.* or putting $r=$ Rad. $S.nA=\frac{nx^{n-1}y}{r^{n-1}}-\frac{\overline{n-1}.\overline{n-2}.yy}{2\cdot 3xx}A-\frac{\overline{n-3}.\overline{n-4}.y^2}{4\cdot 5xx}B-\frac{\overline{n-5}.\overline{n-6}.yy}{6\cdot 7xx}C$, *&c.* where A, B, C, *&c.* are the preceding Terms with their Signs.

Cor. If you ſuppoſe r to be the Diameter, y the Cord of an Arch, x the Cord of the Supplement; then the foregoing Series will be the Cord of the multiple Arch.

PROP. XXV.

The Sine s of an Arch A, *being given; to find the Sine of n* A, *or n times that Arch.*

Let $z=S.nA$. Then by Prop. XI, the Arch belonging to the Sine s, is $s+\frac{s^3}{6rr}+\frac{3s^5}{40r^4}+\frac{5s^7}{112r^6}$, &c. and the Arch belonging to the Sine z is $z+\frac{z^3}{6rr}+\frac{3z^5}{40r^4}+\frac{5z^7}{112r^6}$, &c. and this is n times the former, therefore $z+\frac{z^3}{6r^2}+\frac{3z^5}{40r^4}+\frac{5z^7}{112r^6}$, &c. $=ns+\frac{ns^3}{6rr}+\frac{3ns^5}{40r^4}+\frac{5ns^7}{112r^6}$, &c.

$$\begin{array}{lll}
\text{Now ſuppoſe} & z= & as+bs^3+cs^5+ds^7 \text{ \&c.} \\
\text{then will} & \frac{z^3}{6rr}= & \frac{a^3s^3}{6rr}+\frac{aab}{2rr}s^5+\frac{abb+aac}{2rr}s^7 \text{ \&c.} \\
 & \frac{3z^5}{40r^4}= & \frac{3a^5}{40r^4}s^5+\frac{3a^4b}{8r^4}s^7 \text{ \&c.} \\
 & \frac{5z^7}{112r^6}= & \frac{5a^7}{112r^6}s^7 \text{ \&c.} \\
 & \text{\&c.} &
\end{array}\Bigg\} =$$

$ns+\frac{n}{6rr}s^3+\frac{3ns^5}{40r^4}+\frac{5n}{112r^6}s^7$, &c. then by equating the homologous Powers of s, we have $as=ns$, and $a=n$. Alſo $b+\frac{a^3}{6rr}=\frac{a}{6rr}$, and thence $b=\frac{n-n^3}{6rr}=\frac{n.\overline{1-nn}}{2.3.rr}$. Alſo $c+\frac{aab}{2rr}+\frac{3a^5}{40r^4}=\frac{3n}{40r^4}$, whence $c=\frac{9n-10n^3+n^5}{120r^4}=\frac{n.\overline{1-n^2}.\overline{9-nn}}{2.3.4.5r^4}$. In like manner $d=\frac{225n-259n^3+35n^5-n^7}{5040r^6}=\frac{n.\overline{1-nn}.\overline{9-nn}.\overline{25-nn}}{2.3.4.5.6.7r^6}$ &c. whence $S.nA=ns-\frac{n.\overline{nn-1}}{2.3r^2}s^3+\frac{n.\overline{nn-1}.\overline{nn-3^2}}{2.3.4.5r^4}s^5-\frac{n.\overline{nn-1}.\overline{nn-3^2}.\overline{nn-5^2}}{2.3.4.5.6.7r^6}s^7$, &c. that is

S.nA

$$S.nA = ns - \frac{nn-1}{2.3r^2}s^2A - \frac{nn-3^2}{4.5r^2}s^2B - \frac{n^2-5^2}{6.7r^2}s^2C - \frac{n^2-7^2}{8.9r^2}s^2D,\ \&c.$$ where A, B, C, are the preceding Terms with their Sines. Therefore if n be an odd Number the Series will be finite.

Cor. 1. Since the Coſine of $A = \sqrt{rr-ss} = r - \frac{ss}{2r} - \frac{s^4}{8r^3} - \frac{s^6}{16r^5}$, &c. therefore dividing the Series above by this Series, and then multiplying by $\sqrt{rr-ss}$, and you will have

$$S.A = \frac{ns}{r} - \frac{n^2-2^2}{2.3r^2}s^2A - \frac{n^2-4^2}{4.5r^2}s^2B - \frac{n^2-6^2}{6.7r^2}s^2C - \frac{n^2-8^2}{8.9r^2}s^2D,\ \&c. \times \sqrt{rr-ss}.$$ And therefore if n be an even Number the Series will be finite.

Cor. 2. If s be the Cord of an Arch, x the Cord of the Supplement $d =$ Diameter, then

$$\textit{Cord of } nA = ns - \frac{nn-1}{2.3dd}ssA - \frac{n^2-3^2}{4.5.d^2}ssB - \frac{n^2-5^2}{6.7d^2}ssC - \frac{n^2-7^2}{8.9d^2}ssD,\ \&c.$$ or

$$\textit{Cord of } n.\textit{Arch} = \frac{nsx}{d} - \frac{n^2-2^2}{2.3d^2}s^2A - \frac{n^2-4^2}{4.5.d^2}ssB - \frac{n^2-6^2}{6.7d^2}s^2C - \&c.$$

SCHOLIUM.

If z the Sine of an Arch was given, and the Sine s, of the n^{th} Part of the Arch, was required; you will have by Reverſion of Series,

$$s = \frac{z}{n} + \frac{n^2-1.z^2}{2.3r^2n^2}A + \frac{3n^2-1}{4.5r^2n^2}Bz^2 + \frac{5n^2-1}{6,7r^2n^2}Cz^2\ \&c.$$

PROP.

PROP. XXVI.

The Sine and Coſine of an Arch being given; to find the Coſine of n times that Arch.

Let A = Arch, y = Sine, x = Coſine. Radius $=1$. Then by Cor. 7. Pr. III. $2x\times$ Coſ. nA = Coſ. $\overline{n-1}$.A + Coſ. $\overline{n+1}$.A, therefore ſince $xx+yy=1$, Coſ. $\overline{n+1}$.A $=2x\times$ Coſ. nA $-\overline{xx+yy}\times$ Coſ. $\overline{n-1}$.A: Therefore if you multiply any Coſine by $2x$, and from it ſubtract the foregoing one multiply'd into $xx+yy$, the remainder is the next ſucceeding Coſine. Hence

Coſ. 0A $=r$
Coſ. 1A $=x$
Coſ. 2A $=xx-yy$
Coſ. 3A $=x^3-3xyy$
Coſ. 4A $=x^4-6x^2y^2+y^4$
Coſ. 5A $=x^5-10x^3y^2+5xy^4$
Coſ. 6A $=x^6-15x^4y^2+15x^2y^4-y^6$
Coſ. 7A $=x^7-21x^5y^2+35x^3y^4-7xy^6$.

&c. therefore in general, let the

$$\text{Coſ. }\overline{n-1}.\text{A}=x^{n-1}-ax^{n-3}y^2+bx^{n-5}y^4-cx^{n-7}y^6\ \&c.$$
$$\text{Coſ. }n\text{A}\ =x^{n}-\underset{1}{a}x^{n-2}y^2+\underset{1}{b}x^{n-4}y^4-\underset{1}{c}x^{n-6}y^6\ \&c.$$

Now if the latter be multiply'd by $2x$, and the former by $xx+yy$, and this Product ſubtracted from the firſt, you will have

$$\left.\begin{array}{l} 2x^{n+1}-2\underset{1}{a}x^{n-1}y^2+2\underset{1}{b}x^{n-3}y^4-2\underset{1}{c}x^{n-5}y^6\ \&c. \\ -x^{n+1}+ax^{n-1}y^2-bx^{n-3}y^4+cx^{n-5}y^6 \\ \qquad -x^{n-1}y^2+ax^{n-3}y^4-bx^{n-5}y^6 \end{array}\right\}$$

$=x^{n+1}-\underset{2}{a}x^{n-1}y^2+\underset{2}{b}x^{n-3}y^4-\underset{2}{c}x^{n-5}y^6$ &c. = Coſ. $\overline{n+1}$.A. Then comparing the Coefficients we have $\underset{2}{a}=2\underset{1}{a}-a+1$, that is, by the Method of Increments $a+2\underset{\cdot}{a}+\underset{\cdot\cdot}{a}=2a+2\underset{\cdot}{a}-a+1$, or $\underset{\cdot\cdot}{a}=1=\underset{\cdot}{n}$, therefore $\underset{\cdot}{a}=$ n, the

FIG.

n, the Integral, and $\underset{1}{a} = \frac{n\underset{-1}{n}}{2}$, for when $n=2$, $\underset{1}{a}$ or $\frac{n\underset{-1}{n}}{2} = 1$. Likewiſe $b = 2\underset{2}{b} - \underset{1}{b} + a$, and $\underset{\cdot\cdot}{b} = a = \frac{\underset{-2}{n}\,\underset{-1}{n}}{2}$, and $\underset{\cdot}{b} = \frac{\underset{-3}{n}\,\underset{-2}{n}\,\underset{-1}{n}}{2\cdot 3}$, and $b = \frac{\underset{-4}{n}\,\underset{-3}{n}\,\underset{-2}{n}\,\underset{-1}{n}}{2\cdot 3\cdot 4}$, (for when $n=5$, $b=1$.) and $\underset{1}{b} = \frac{\underset{-3}{n}\,\underset{-2}{n}\,\underset{-1}{n}n}{2\cdot 3\cdot 4}$. Again $\underset{\cdot\cdot}{c} = b = \frac{\underset{-4}{n}\ldots\underset{-1}{n}}{2\cdot 3\cdot 4}$, $\underset{\cdot}{c} = \frac{\underset{-5}{n}\ldots\underset{-1}{n}}{2\cdot 3\cdot 4\cdot 5}$ and $c = \frac{\underset{-6}{n}\ldots\underset{-1}{n}}{2\ldots 6}$, and therefore $\underset{1}{c} = \frac{\underset{-5}{n}\ldots n}{2\ldots 6}$. Likewiſe $\underset{1}{d} = \frac{\underset{-7}{n}\ldots n}{2\ldots 7}$, &c. then ſince $\underset{\cdot}{n} = 1$, therefore n, $\underset{-1}{n}$, $\underset{-2}{n}$, &c. are equal to n, $n-1$, $n-2$, &c. whence putting r for Radius,

$$Coſ.\, nA = \frac{1}{r^{n-1}} \times : x^n - \frac{n\cdot\overline{n-1}}{2} x^{n-2}y^2 + \frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}}{2\cdot 3\cdot 4} x^{n-4}y^4 - \frac{n\cdot\overline{n-1}\cdot\overline{n-2}\cdot\overline{n-3}\cdot\overline{n-4}\cdot\overline{n-5}}{2\cdot 3\cdot 4\cdot 5\cdot 6} x^{n-6}y^6.$$

&c. or

$$Coſ.\, nA = \frac{x^n}{r^{n-1}} - \frac{n : \overline{n-1}}{1\cdot 2} A\frac{yy}{xx} - \frac{\overline{n-2}\cdot\overline{n-3}}{3\cdot 4} B\frac{yy}{xx} - \frac{n-4\cdot\overline{n-5}}{5\cdot 6} C\frac{yy}{xx} - \frac{\overline{n-6}\cdot\overline{n-7}}{7\cdot 8} D\frac{yy}{xx},$$

&c. where A, B, C, &c. are the foregoing Terms with their Signs.

Cor. If you ſuppoſe r the Diameter, y the Cord of an Arch, x the Cord of the Supplement, then the foregoing Series will be the Cord of the Supplement of the multiple Arch.

PROP.

PROP. XXVII.

The Sine s of an Arch a being given; to find the Coſine of n a, or n times that Arch.

Let $z =$ Coſ. na, Rad. $= 1$, then by Cor. 1. Pr. 12.

$$z = 1 - \frac{n^2a^2}{2} + \frac{n^4a^4}{24} - \frac{n^6a^6}{720} + \frac{n^8a^8}{40320} \text{ \&c.}$$

but by Prop. 11.

$$a = s + \frac{s^3}{6} + \frac{3s^5}{40} + \frac{5s^7}{112} \text{ \&c.}$$

whence

$$\begin{array}{lllll} aa = & ss + \frac{s^4}{3} & + \frac{8s^6}{45} & + \frac{4s^8}{35} & \text{\&c.} \\ a^4 = & s^4 & + \frac{2}{3}s^6 & + \frac{7}{15}s^8 & \text{\&c.} \\ a^6 = & & s^6 & + s^8 & \\ a^8 = & & & s^8 & \\ \text{\&c.} & & & & \end{array}$$

then all theſe Values of the Powers of a being ſubſtituted in the Value of z, gives

$$\begin{array}{lllll} z = 1 - \frac{nnss}{2} & - \frac{n^2s^4}{6} & - \frac{4n^2s^6}{45} & - \frac{2n^2s^8}{35} & \text{\&c.} \\ & + \frac{n^4s^4}{24} & + \frac{n^4s^6}{36} & + \frac{7n^4s^8}{360} & \\ & & - \frac{n^6s^6}{720} & - \frac{n^6s^8}{720} & \\ & & & + \frac{n^8s^8}{40320} & \text{\&c.} \end{array}$$

whence

$$z = 1 - \frac{n^2}{2}s^2 + \frac{nn.\overline{n^2 - 2^2}}{2.3.4}s^4 - \frac{n^2.\overline{n^2 - 2^2}.\overline{n^2 - 4^2}}{2.3.4.5.6}s^6 \text{ \&c.}$$

that is if $r =$ Radius.

Coſ. $na = r - \frac{n^2s^2}{1.2r^2}A - \frac{n^2 - 2^2}{3.4r^2}ssB - \frac{n^2 - 4^2}{5.6r^2}ssC - \frac{n^2 - 6^2}{7.8r^2}ssD$ &c. where A, B, C, &c. are the foregoing Terms with their Signs. Therefore if n be an even Number, the Series will be finite.

Cor. Since $\sqrt{rr-ss}=r-\frac{ss}{2r}-\frac{s^4}{8r^3}-\frac{s^6}{16r^5}-\frac{5s^8}{128r^7}$ *&c.* $=$ Coſ. A. therefore dividing the foregoing Series, by this; and then multiply by $\sqrt{rr-ss}$, and you will have *Coſ.* $na=\sqrt{rr-ss}\times:1-\frac{n^2-1}{1\cdot 2r^2}ssA-\frac{n^2-3^2}{3\cdot 4r^2}ssB-\frac{n^2-5^2}{5\cdot 6r^2}ssC-\frac{n^2-7^2}{7\cdot 8r^2}ssD$ *&c.* and therefore if n be an odd Number the Series will be finite.

PROP. XXVIII.

The Cord of an Arch, and the Cord of its Supplement being given; to find the Cord of n times that Arch.

Let A be the Arch, y the Cord, x the Cord of the Supplement, Radius $=1$, then by Cor. Pr. XXIII. $x\times$ Cord. nA$=$ Cord. $\overline{n-1}$. A$+$ Cord. $\overline{n+1}$. A, and Cord. $\overline{n+1}$. A$=x\times$Cord. nA$-$Cord. $\overline{n-1}$. A. that is, if any Cord be multiply'd by x, and the foregoing one ſubtracted, will give the next following Cord. whence

Cord. A$=y$
Cord. 2A$=xy$
Cord. 3A$=x^2y-y$
Cord. 4A$=x^3y-2xy$
Cord. 5A$=x^4y-3x^2y+y$, *&c.*

And in general let the Cord of, nA be $=yx^{n-1}-ax^{n-3}y+bx^{n-5}y-cx^{n-7}y$ *&c.* then will

$$\left.\begin{array}{l} yx^n-ayx^{n-2}+byx^{n-4}-cyx^{n-6}\ \&c. \\ \quad -yx^{n-2}+\underset{-1}{a}yx^{n-4}-\underset{-1}{b}yx^{n-6}\ \&c. \end{array}\right\}=$$

$$yx^n-\underset{1}{a}yx^{n-2}+\underset{1}{b}yx^{n-4}-\underset{1}{c}yx^{n-6}\ \&c.=\text{Cord.}\,\overline{n+1}\,.\,\text{A}.$$

then equating the Coefficients $\underset{1}{a}=a+1$, and $\underset{1}{a}-a$ or $a=1=n$, and the Integral is $\underset{-2}{a=n}$ (for when $n=3$, a

or $\underset{-2}{n}=1$). Again, $\underset{1}{b}$ or $b+\underset{\cdot}{b}=b+\underset{-1}{a}$, or $\underset{\cdot}{b}=\underset{-1}{a}=\underset{-3}{n}$, and FIG. $b=\frac{\overset{n}{-3}\overset{n}{-4}}{2}$, alſo $\underset{1}{c}=c+\underset{-1}{b}$, or $\underset{\cdot}{c}=\underset{-1}{b}=\frac{\overset{n}{-4}\overset{n}{-5}}{2}$, and $c=\frac{\overset{n}{-4}\overset{n}{-5}\overset{n}{-6}}{2\cdot 3}$. In like manner $d=\frac{\overset{n}{-5}\overset{n}{-6}\overset{n}{-7}\overset{n}{-8}}{2\cdot 3\cdot 4}$, &c. whence the *Cord* .nA$=yx^{n-1}-\frac{n-2}{1}yx^{n-3}+\frac{\overline{n-3}.\overline{n-4}}{1.2}yx^{n-5}-\frac{\overline{n-4}.\overline{n-5}.\overline{n-6}}{1.2.3}yx^{n-7}+\frac{\overline{n-5}.\overline{n-6}.\overline{n-7}.\overline{n-8}}{1.2.3.4}yx^{n-9}$ &c. that is putting Radius $=r$, then

$$Cord.\,n\,Arch=\frac{yx^{n-1}}{r^{n-1}}-\frac{n-2}{1}A\frac{rr}{xx}-\frac{\overline{n-3}.\overline{n-4}}{2.\overline{n-2}}B\frac{rr}{xx}-\frac{\overline{n-5}.\overline{n-6}}{3.\overline{n-3}}C\frac{rr}{xx}-\frac{\overline{n-7}.\overline{n-8}}{4.\overline{n-4}}D\frac{rr}{xx}-\frac{\overline{n-9}\,\overline{n-10}}{5.\overline{n-5}}E\frac{rr}{xx}$$

&c.

Cor. If $r=$ Radius, $y=$ Sine, $x=$ Coſine of an Arch, then *the Sine of the* n^{th} *Arch* $=\frac{2x^{n-1}}{r^{n-1}}y-\frac{\overline{n-2}.rr}{4xx}A-\frac{\overline{n-3}.\overline{n-4}.rr}{4.2.\overline{n-2}.xx}B-\frac{\overline{n-5}.\overline{n-6}.rr}{4.3.\overline{n-3}.xx}C-\frac{\overline{n-7}.\overline{n-8}.rr}{4.4.\overline{n-4}.xx}D$ &c. this is plain by putting $\frac{1}{2}r$ for r in the foregoing Series.

PROP. XXIX.

The Square of the Cord of an Arch being given; to find the Square of the Cord of n times the Arch.

Let a, b, c be the Cords of three Arches in arithmetic Progreſſion, Rad. $=1$, $x=$ Cord of the common Difference, and $\sqrt{4-xx}=$ Cord of the Sup. common Difference. Then by Prop. XXIII. Rad. ×Sum of the Cords of the extream Arches = Cord mean × Cord Sup. common Difference, that is, $b\sqrt{4-xx}=a+c$. And by Cor. 4. Pr. XXII. $ac=bb-xx$

FIG. $-xx$. Whence $4bb-bbxx=a\underline{a+c}c+2ac=aa+cc+2bb-2xx$, therefore $bb+\overline{2-xx}+2xx-aa=cc$, that is, in any Series of the Squares of Cords, if the Square of any Cord be multiply'd by $-xx+2$, and the preceding one ſubtracted, and $2xx$ added, you will have the Square of the next following Cord. Thus, Square of the Cord

$$\begin{aligned}
\text{of } A &= xx\\
2A &= -x^4+4x^2\\
3A &= x^6-6x^4+9x^2\\
4A &= -x^8+8x^6-20x^4+16x^2\\
5A &= x^{10}-10x^8+35x^6-50x^4+25x^2,\\
&\text{\&c.}
\end{aligned}$$

therefore let any one in general be expreſs'd thus,

$$x^{v}-ax^{v-2}+bx^{v-4}-cx^{v-6}\ \text{\&c.}$$

then the next ſucceeding one, (changing the Signs), will be $-\overset{v}{x^{1}}+a\overset{v-2}{x^{1}}-\underset{1}{b}\overset{v-4}{x^{1}}+\underset{1}{c}\overset{v-6}{x^{1}}$ &c. = (by Conſtruction to)

$$\left.\begin{array}{lll} -x^{v+2}+ax^{v} & -bx^{v-2}+ & cx^{v-4}\\ +2x^{v} & -2ax^{v-2}+ & 2bx^{v-4}\\ & +\ x^{v-2}- & \underset{-1}{a}x^{v-4}\end{array}\right\}\text{\&c.}+2xx.$$

then comparing the homologous Terms we have, $\underset{1}{v}=v+2$, or $\underset{1}{v}-v$ that is $\underset{\cdot}{v}=2=2\underset{\cdot}{n}$, and $v=2n$. Again, $\underset{1}{a}=a+2$, or $\underset{1}{a}-a$ or $\underset{\cdot}{a}=2=\underset{\cdot}{v}$, whence $a=v=2n$. Alſo $\underset{1}{b}=b+2a-1$, or $\underset{\cdot}{b}=2a-1=2v-\frac{1}{2}\underset{\cdot}{v}=v\underset{\cdot}{v}-\frac{1}{2}\underset{\cdot}{v}$,

therefore $b=\dfrac{\overset{vv}{-1}}{2}-\dfrac{v}{2}$.

Again, $\underset{1}{c}=c+2\underset{-1}{b}-a$, or $c=\underset{-1}{vv}-v-v=\underset{-1}{vv}-2\underset{-1}{v}+2=\underset{-1}{vv}-2\underset{\cdot}{v}+v$, and $c=\dfrac{\overset{v}{-2}\ \overset{vv}{-1}}{2\cdot 3}-\dfrac{\overset{vv}{-1}}{2}+v$.

In

FIG.

In like manner $\underset{\cdot}{d} = 2c - \underset{-1}{b} = \frac{\underset{-2}{\nu}\;\underset{-1}{\nu\nu}}{3} - \frac{3\,\underset{-1}{\nu\nu}}{2} + \frac{9\nu}{2} - \frac{5\underset{\cdot}{\nu}}{2}$

and $d = \frac{\underset{-3}{\nu}\;\underset{-2}{\nu}\;\underset{-1}{\nu\nu}}{2\cdot3\cdot4} - \frac{\underset{-2}{\nu}\;\underset{-1}{\nu\nu}}{2\cdot2} + \frac{9\,\underset{-1}{\nu\nu}}{2.2.2} - \frac{5\nu}{2}$.

But ſince $\underset{\cdot}{\nu} = 2 \cdot \underset{-1}{\nu} = \nu - 2 \cdot \underset{-2}{\nu} = \overline{\nu - 4}$ &c. we ſhall find

$b = \nu \times \frac{\nu - 3}{2}$, $c = \nu \times \frac{\nu - 4}{2} \times \frac{\nu - 5}{3}$, $d = \nu \times \frac{\overline{\nu - 5} \times \overline{\nu - 6} \times \overline{\nu - 7}}{2\cdot3\cdot4}$

&c. therefore if x is the Cord of an Arch, and b the Cord of n times the Arch, then will

$$x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2} x^{2n-4} - 2n \times \frac{\overline{2n-4}.\overline{2n-5}}{2\cdot3} x^{2n-6}$$

$$+ 2n \times \frac{\overline{2n-5}.\overline{2n-6}.\overline{2n-7}}{2\cdot3\cdot4} x^{2n-8} \text{ \&c.} = \pm bb,$$ *according as n is odd or even; and the Series continued to n Terms.*

Or $\frac{x^{2n}}{r^{2n-1}} - \frac{2nr^2}{xx} \mathrm{A} - \frac{2n-3}{2xx} rr\,\mathrm{B} - \frac{\overline{2n-4}.\overline{2n-5}}{3.\overline{2n-3}.xx} rr\,\mathrm{C}$

$- \frac{\overline{2n-6}.\overline{2n-7}}{4.\overline{2n-4}.xx} rr\,\mathrm{D} - \frac{\overline{2n-8}.\overline{2n-9}}{5,\overline{2n-5}.xx} rr\,\mathrm{E} -$

$\frac{\overline{2n-10}.\overline{2n-11}}{6.\overline{2n-6}.xx} rr\,\mathrm{F} -$ &c. $= \pm bb$.

Otherwiſe.

Since the Law of the Continuation of the Series of the Squares of the Cords is this, that any Cord multiply'd by $2 - xx$, and the foregoing one ſubtracted, and $2xx$ added, gives the next. Therefore

□ 1 Cord $= xx$
$2. = 4x^2 - x^4$
$3. = 9x^2 - 6x^4 + x^6$
$4. = 16x^2 - 20x^4 + 8x^6 - x^8$
$5. = 25x^2 - 50x^4 + 35x^6 - 10x^8 + x^{10}$
&c.

FIG. and in general $ax^2 - bx^4 + cx^6 - dx^8$ &c. = Square of the Cord of the n^{th} Arch. Whence by the Conſtruction,

$$\left.\begin{array}{l} 2\underset{1}{a}x^2 - 2\underset{1}{b}x^4 + 2\underset{1}{c}x^6 - 2\underset{1}{d}x^8 \\ +2 \quad - \underset{1}{a} \quad + \underset{1}{b} \quad - \underset{1}{c} \\ -a \quad + b \quad - c \quad + d \end{array}\right\} = \underset{2}{a}x^2 - \underset{2}{b}x^4 + \underset{2}{c}x^6 - \underset{2}{d}x^8 \text{ \&c.}$$

Here then $\underset{2}{a} = 2\underset{1}{a} - a + 2$, that is by the Method of Increments $a + 2\underset{\cdot}{a} + \underset{\cdot\cdot}{a} = 2a + 2\underset{\cdot}{a} - a + 2$, that is $\underset{\cdot\cdot}{a} = 2 = 2\underset{\cdot}{n}$, and $\underset{\cdot}{a} = 2n + A$, and $a = \frac{2n\underset{-1}{n}}{2} + An$, and when $\underset{\cdot}{n} = 1$, $A = 1$, therefore $a = \underset{-1}{n}n + n$.

Again, $\underset{2}{b} = 2\underset{1}{b} + \underset{1}{a} - b$, and their Increments $b + 2\underset{\cdot}{b} + \underset{\cdot\cdot}{b} = 2b + 2\underset{\cdot}{b} + a + \underset{\cdot}{a} - b$, and $\underset{\cdot\cdot}{b} = a + \underset{\cdot}{a}$, likewiſe $\underset{\cdot\cdot}{c} = b + \underset{\cdot}{b}$, $\underset{\cdot\cdot}{d} = c + \underset{\cdot}{c}$ and ſo on; that is, $\underset{\cdot\cdot}{b} = \underset{1}{a}$, $\underset{\cdot\cdot}{c} = \underset{1}{b}$, $\underset{\cdot\cdot}{d} = \underset{1}{c}$ &c. therefore $\underset{\cdot\cdot}{b} = n\underset{1}{n} + \underset{1}{n}$, and $\underset{\cdot}{b} = \frac{\underset{-1}{n}n\underset{1}{n}}{3} + \frac{n\underset{1}{n}}{2}$, and $b = \frac{\underset{-2}{n}\underset{-1}{n}n\underset{1}{n}}{3\cdot4} + \frac{\underset{-1}{n}n\underset{1}{n}}{2\cdot3}$. Again, $\underset{\cdot}{c} = \frac{\underset{-1}{n}n\underset{1}{n}\underset{2}{n}}{3\cdot4} + \frac{n\underset{1}{n}\underset{2}{n}}{2\cdot3}$, and $c = \frac{\underset{-2}{n}\underset{-1}{n}n\underset{1}{n}\underset{2}{n}}{3\cdot4\cdot5} + \frac{\underset{-1}{n}n\underset{1}{n}\underset{2}{n}}{2\cdot3\cdot4}$; and $c = \frac{\underset{-3}{n}\ldots\underset{2}{n}}{3\ldots6} + \frac{\underset{-2}{n}..\underset{2}{n}}{2..5}$. In like manner $d = \frac{\underset{-4}{n}\ldots\underset{3}{n}}{3\ldots8} + \frac{\underset{-3}{n}..\underset{3}{n}}{2..7}$, and $e = \frac{\underset{-5}{n}..\underset{4}{n}}{3..10} + \frac{\underset{+4}{n}..\underset{3}{n}}{2..9}$, and ſo on. Then ſince $\underset{\cdot}{n} = 1$, and n, $\underset{-1}{n}$, $\underset{1}{n}$, $\underset{-2}{n}$, $\underset{2}{n}$ &c. $= n$, $n-1$, $n+1$, $n-2$, $n+2$ &c. reſpectively. Therefore by Reduction will be found $a = nn$, $b = \frac{nn.nn-1}{2.2.3}$,

c=

$c = \frac{nn.\overline{nn-1}.\overline{nn-4}}{3.1.2.3.4.5}$, $d = \frac{nn.\overline{nn-1}.\overline{nn-4}.\overline{nn-9}}{4.1\ldots.7}$ &c. *whence the Square of the Cord of the* n^{th} *Arch* $= nnx^2 - \frac{nn.\overline{nn-1}}{2.1.2.3}x^4 + \frac{nn.\overline{nn-1}.\overline{nn-4}}{3.1.2.3.4.5}x^6 - \frac{nn.\overline{nn-1}.\overline{nn-4}.\overline{nn-9}}{4.1.2.3.4.5.6.7}x^8$ &c. or putting $r =$ Radius, □ Cord n^{th} Arch $= nnx^2 - \frac{\overline{nn-1}.x^2}{2.1.2.3r^2}A - \frac{2.\overline{nn-4}.x^2}{3.4.5.rr}B - \frac{3.\overline{nn-3^2}}{4.6.7r^2}x^2C - \frac{4.\overline{n^2-4^2}.x^2}{5.8.9r^2}D - \frac{5.\overline{n^2-5^2}.x^2}{6.10.11.r^2}E$ &c.

Cor. 1. If x be the Cord of an Arch, Radius $= r$, and c the Cord of n times the Arch; then $\frac{x^n}{r^{n-1}} - \frac{nx^{n-2}}{r^{n-3}} + n \times \frac{\overline{n-3}.x^{n-4}}{2r^{n-5}} - n \times \frac{\overline{n-4}.\overline{n-5}.x^{n-6}}{2.3r^{n-7}} + n \times \frac{\overline{n-5}.\overline{n-6}.\overline{n-7}}{2.3.4r^{n-9}}x^{n-8} - n \times \frac{\overline{n-6}.\overline{n-7}.\overline{n-8}\,\overline{n-9}}{2.3.4.5r^{n-11}}x^{n-10}$ &c. $= \pm c$, according as $\frac{n+1}{2}$ is odd or even, the Series continued to $\frac{n+1}{2}$ Terms: that is, $+c$ when odd, and $-c$ when even.

Or thus; $\sqrt{4rr-xx} : \times$ into $\frac{x^{n-1}}{r^{n-1}} - \frac{\overline{n-2}.x^{n-3}}{r^{n-3}} + \frac{\overline{n-3}.\overline{n-4}}{1.2r^{n-5}}x^{n-5} - \frac{\overline{n-4}.\overline{n-5}.\overline{n-6}}{1.2.3r^{n-7}}x^{n-7} + \frac{\overline{n-5}.\overline{n-6}.\overline{n-7}.\overline{n-8}}{1.2.3.4r^{n-9}}x^{n-9} -$ &c. $= \pm c$, according as $\frac{1}{2}n$ is odd or even, and the Series continu'd to $\frac{n}{2}$ Terms.

For the former Series is the Root of $x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2}x^{2n-4}$ &c. the Square of the Cord, found above by the firſt Method. And the latter Series is found by dividing the Series $x^{2n}-2nx^{2n-2}$ &c. by $-xx+4$ (or rather by $xx-4$), and extracting the Root of the Quotient.

Cor. 2. If v be the verſed Sine of an Arch, V the verſed ſine of n times the Arch, then will

$$\frac{\overline{2v}^{n}}{2r^{n}}-\frac{2nr}{2v}A-\frac{\overline{2n-3}.r}{2.2v}B-\frac{\overline{2n-4}.\overline{2n-5}.r}{3.\overline{2n-3}.2v}C-\frac{\overline{2n-6}.\overline{2n-7}.r}{4.\overline{2n-4}.2v}D-\frac{\overline{2n-8}.\overline{2n-9}.r}{5.\overline{2n-5}.2v}E\text{ \&c.}=\underline{\pm}V,$$

according as n is odd or even, the Series continued to n Terms.

$$\textit{Or } V=n^2v-\frac{\overline{n^2-1}.2v}{2.2.3r}A-\frac{2.\overline{n^2-2^2}.2v}{3.4.5r}B-\frac{3.\overline{n^2-3^2}.2v}{4.6.7r}C-\frac{4.\overline{n^2-4^2}.2v}{5.8.9r}D-\text{\&c.}$$

This appears by ſubſtituting $2rv$ for xx, and $2rV$ for bb, to which they are equal, by Cor. 8. Prop. I.

PROP. XXX.

The Cord of the Supplement of an Arch being given; to find the Cord of the Supplement of n times the Arch.

Let Radius $=r=1$. $x=$ Cord of the Supplement, A=Arch; then by Cor. Prop. XXIII. $1 : x ::$ Cord. Sup. nA :: Cord. Sup. $\overline{n-1}$.A + Cord. Sup. $\overline{n+1}$.A. whence Cord. Sup. $\overline{n+1}$. A $=x\times$ Cord. Sup. nA − Cord. Sup. $\overline{n-1}$. A, that is, if the Cord of the Sup. of any Arch, in a Series thereof, be multiply'd by x, and the preceding Cord of the Sup. ſubtracted, you will have the next ſucceeding one. Hence

Cord Sup. $0A=2$
$1A=x$
$2A=x^2-2$
$3A=x^3-3x$
$4A=x^4-4x^2+2$
$5A=x^5-5x^3+5x$
$6A=x^6-6x^4+9x^2-2$
&c.

and in general $nA=x^n-ax^{n-2}+bx^{n-4}-cx^{n-6}$ &c.
then $\overline{n+1}.A=x^{n+1}-\underset{1}{a}x^{n-1}+\underset{1}{b}x^{n-3}-\underset{1}{c}x^{n-5}$ &c.

$$=\left\{\begin{array}{l} x^{n+1}-ax^{n-1}+bx^{n-3}-cx^{n-5} \\ \qquad -1x^{n-1}+\underset{-1}{a}x^{n-3}-\underset{-1}{b}x^{n-5} \end{array}\right\}$$ &c. by Conſtruction. Hence $\underset{1}{a}=a+1$, and $a=1=n$, whence $a=n$.

Alſo $\underset{1}{b}=b+\underset{-1}{a}$, and $b=\underset{-1}{a}=\underset{-1}{n}$, whence $b=\frac{\overset{n}{-2}\,\overset{n}{-1}}{2}-1$, for when $n=4$, $b=2$. therefore $b=\frac{\overset{n}{-3}\,\overset{n}{-2}}{2}+\underset{-2}{n}-1=\frac{\overset{n}{-3}\,\overset{n}{-2}}{2}+\underset{-3}{n}=\frac{\overset{n}{}\overset{n}{-3}}{2}$.

Again, $\underset{1}{c}=c+\underset{-1}{b}$, and $c=\underset{-1}{b}=\frac{\overset{n}{-4}\,\overset{n}{-1}}{2}+\underset{-4}{n}$, and $c=\frac{\overset{n}{-3}\,\overset{n}{-4}\,\overset{n}{-5}}{2\cdot 3}+\frac{\overset{n}{-4}\,\overset{n}{-5}}{2}=\frac{\overset{n}{}\,\overset{n}{-4}\,\overset{n}{-5}}{1\cdot 2\cdot 3}$.

Again, $d=\underset{-1}{c},=\frac{\overset{n}{-4}\,\overset{n}{-5}\,\overset{n}{-6}}{2\cdot 3}+\frac{\overset{n}{-5}\,\overset{n}{-6}}{2}$, and $d=\frac{\overset{n}{-4}\,\overset{n}{-5}\,\overset{n}{-6}\,\overset{n}{-7}}{2\cdot 3\cdot 4}+\frac{\overset{n}{-5}\,\overset{n}{-6}\,\overset{n}{-7}}{2\cdot 3}=\frac{\overset{n}{}\,\overset{n}{-5}\,\overset{n}{-6}\,\overset{n}{-7}}{1\cdot 2\cdot 3\cdot 4}$. In like manner $e=\frac{\overset{n}{}\,\overset{n}{-6}\,\overset{n}{-7}\,\overset{n}{-8}\,\overset{n}{-9}}{2\cdot 3\cdot 4\cdot 5}$ &c. whence

The

FIG. *The Cord of the Supplement of the* n^{th} *Arch is* $= x^n - nx^{n-2} + \frac{n.\overline{n-3}}{1.2}x^{n-4} - \frac{n.\overline{n-4}.\overline{n-5}}{1.2.3}x^{n-6} + \&c. = \frac{x^n}{r^{n-1}} - \frac{n.rr}{xx}A - \frac{\overline{n-3}.rr}{2xx}B - \frac{\overline{n-4}.\overline{n-5}.rr}{3.\overline{n-3}.xx}C - \frac{\overline{n-6}.\overline{n-7}.rr}{4.\overline{n-4}.xx}D - \frac{\overline{n-8}.\overline{n-9}.rr}{5.\overline{n-5}.xx}E - \&c.$ continued to $\frac{n+1}{2}$ terms when n is odd, or to $\frac{n+2}{2}$ terms when even.

Cor. Hence if Radius $=r$. $x=$ Coſine of an Arch, then *the Coſine of the* n^{th} *Arch* $= \frac{2x^n}{2r^{n-1}} - \frac{nrr}{4.xx}A - \frac{\overline{n-3}.rr}{4.2xx}B - \frac{\overline{n-4}.\overline{n-5}.rr}{4.3.\overline{n-3}.xx}C - \frac{\overline{n-6}.\overline{n-7}rr}{4.4.\overline{n-4}.xx}D - \&c.$ continued to $\frac{n+1}{2}$ terms when n is odd, or to $\frac{1}{2}n+1$ terms when even.

This follows from the foregoing Series putting $\frac{1}{2}r$ for r.

PROP. XXXI.

Given the Cord of an Arch, and the Cord of the Supplement of twice the Arch; to find the Cord of $2m+1$ *times the Arch.*

Let Radius $=r=1$. Arch $=A$. Cord of $A=y$, Cord Sup. of $2A=z$. Then by Cor. Pr. XXIII. $1 : z ::$ Cord of $\overline{2m+1}.A$: Cord of $\overline{2m-1}.A +$ Cord of $\overline{2m+3}.A$, therefore Cord of $\overline{2m+3}.A = z \times$ Cord of $\overline{2m+1}.A -$ Cord of $\overline{2m-1}.A$, that is in a Series of theſe Cords, if any one be multiply'd by z, and the foregoing one ſubtracted, you will have the next following one. Whence

Cord $A=y$

$3A=z+1:\times y$

$5A=z^2+z-1\times y$

$7A=z^3+z^2-2z-1\times y$

$9A=z^4+z^3-3z^2-2z+1\times y$

$11A=z^5+z^4-4z^3-3z^2+3z+1\times y$

$13A=z^6+z^5-5z^4-4z^3+6z^2+3z-1\times y$

$15A=z^7+z^6-6z^5-5z^4+10z^3+6z^2-4z-1\times y$

and in general Cord $\overline{2m+1}.A=z^m+z^{m-1}-az^{m-2}-bz^{m-3}+cz^{m-4}+dz^{m-5}-ez^{m-6}-fz^{m-7}$ *&c.* then by Conſtruction,

$$\left\{\begin{array}{l} z^{m+1}+z^m-az^{m-1}-bz^{m-2}+cz^{m-3}+dz^{m-4}-ez^{m-5}\ \&c. \\ \qquad\qquad -z^{m-1}-z^{m-2}+\underset{-1}{a}z^{m-3}+\underset{-1}{b}z^{m-4}-\underset{-1}{c}z^{m-5} \end{array}\right.$$

$=z^{m+1}+z^m-\underset{1}{a}z^{m-1}-\underset{1}{b}z^{m-2}+\underset{1}{c}z^{m-3}+\underset{1}{d}z^{m-4}-\underset{1}{e}z^{m-5}$ $\underset{1}{\&c.}$ = Cord of $\overline{2m+3}.A.$ whence $\underset{1}{a}=a+1$, and $\underset{1}{a}-a$ or $\underset{.}{a}=1=\underset{.}{m}$, whence $a=\underset{-1}{m}$, for when $m=2$, $a=1$.

Again, $\underset{1}{b}-b$ or $\underset{.}{b}=1=\underset{.}{m}$, and $b=\underset{-2}{m}$, for when $m=3$, $b=1$.

Alſo $\underset{1}{c}=c+\underset{-1}{a}$, and $\underset{1}{c}-c$ or $\underset{.}{c}=\underset{-2}{m}$, whence $c=\frac{\overset{m}{-3}\ \overset{m}{-2}}{2}$.

Likewiſe $\underset{1}{d}-d$ or $\underset{.}{d}=\underset{-1}{b}=\underset{-3}{m}$, and $d=\frac{\overset{m}{-4}\ \overset{m}{-3}}{2}$.

Alſo $\underset{.}{e}=\underset{-1}{c}=\frac{\overset{m}{-4}\ \overset{m}{-3}}{2}$, and $e=\frac{\overset{m}{-5}\ \overset{m}{-4}\ \overset{m}{-3}}{2\cdot 3}$, likewiſe $\underset{.}{f}=\underset{-1}{d}$, and $f=\frac{\overset{m}{-6}\ \overset{m}{-5}\ \overset{m}{-4}}{2\cdot 3}$, $g=\frac{\overset{m}{-7}\ \overset{m}{-6}\ \overset{m}{-5}\ \overset{m}{-4}}{2\cdot 3\cdot 4}$, $h=\frac{\overset{m}{-8}\ \overset{m}{-7}\ \overset{m}{-6}\ \overset{m}{-5}}{2\cdot 3\cdot 4}$, $i=\frac{\overset{m}{-9}\ \ldots\ \overset{m}{-5}}{2\cdot 3\cdot 4\cdot 5}$, $k=\frac{\overset{m}{-10}\ \ldots\ \overset{m}{-6}}{2\cdot 3\cdot 4\cdot 5}$ *&c.* whence the

Cord of the $\overline{2m+1}$. *Arch* $=y\times$ into $z^m+z^{m-1}\underset{m-1}{-}$

$$\frac{m-1}{1}z^{m-2}-\frac{m-2}{1}z^{m-3}+\frac{\overline{m-2}\,,\,\overline{m-3}}{2}z^{m-4}+$$
$$\frac{\overline{m-3}\cdot\overline{m-4}}{2}z^{m-5}-\frac{\overline{m-3}\cdot\overline{m-4}\cdot\overline{m-5}}{2\cdot 3}z^{m-6}-$$
$$\frac{\overline{m-4}\cdot\overline{m-5}\cdot\overline{m-6}}{2\cdot 3}z^{m-7}+\frac{\overline{m-4}\ldots\ldots\overline{m-7}}{2\cdot 3\,;\,4}z^{m-8}+$$
$$\frac{\overline{m-5}\ldots\ldots\overline{m-8}}{2\cdot 3\cdot 4}z^{m-9}-\frac{\overline{m-5}\ldots\ldots\overline{m-9}}{2\cdot 3\cdot 4\cdot 5}z^{m-10}-$$
$$\frac{\overline{m-6}\ldots\ldots\overline{m-10}}{2\cdot 3\cdot 4\cdot 5}z^{m-11}$$

&c. continued to $m+1$ Terms.

PROP. XXXII.

Given the Cord of the Supplement of an Arch, and the Cord of the Supplement of twice the Arch; to find the Cord of the Supplement of $\overline{2m+1}$ *times the Arch.*

Let Radius $=1$, Arch $=A$, Cord Sup. $A=x$, Cord Sup. $2A=z$, then as in the laſt Prop. Cord. Sup. $\overline{2m+3}.A=z\times$ Cord . Sup. $\overline{2m+1}.A-$ Cord . Sup. $\overline{2m-1}.A$. that is, if, in a Series, any one be multiply'd by z, and the foregoing ſubtracted, you will have the next following one. Hence Cord . Sup.

of $A=x$
of $3A=z-1 : \times x$
of $5A=z^2-z-1\times x$
$7A=z^3-z^2-2z+1\times x$
$9A=z^4-z^3-3z^2+2z+1\times x$
$11A=z^5-z^4-4z^3+3z^2+3z-1\times x$
&c.

And in general, Cord . Sup. $\overline{2m+1}.A=z^m-z^{m-1}-az^{m-2}+bz^{m-3}+cz^{m-4}-dz^{m-5}\pm ez^{m-6}$ &c. therefore by Conſtruction

$$\left.\begin{array}{r}z^{m+1}-z^m-az^{m-1}+bz^{m-2}+cz^{m-3}\\ -1z^{m-1}+1z^{m-2}+az^{m-3}\\ -1\end{array}\right\}\text{\&c.}=$$

$$z^{m+1}-z^m-\underset{1}{a}z^{m-1}+\underset{1}{b}z^{m-2}+\underset{1}{c}z^{m-3}=\text{Cord . Sup.}$$

$\overline{2m+3}$.A. hence $a_1=a+1$, $b_1=b+1$, $c_1=c+a_{-1}$, $d_1=d+b_{-1}$ &c. all as in the laſt Prop. and therefore the Coefficients a, b, c, &c. will be the very ſame, whence the

$$\textit{Cord. Sup. } \overline{2m+1}.A = x \times \text{ into } z^m - z^{m-1} - \frac{m-1}{1} z^{m-2} + \frac{m-2}{1} z^{m-3} + \frac{\overline{m-2}.\overline{m-3}}{2} z^{m-4} - \frac{\overline{m-3}.\overline{m-4}}{2} z^{m-5} - \frac{\overline{m-3}.\overline{m-4}.\overline{m-5}}{2.3} z^{m-6} + \frac{\overline{m-4}.\overline{m-5}.\overline{m-6}}{2.3} z^{m-7} + \frac{\overline{m-4}\ldots\overline{m-7}}{2.3.4} z^{m-8} \text{ \&c.}$$

Cor. If Rad. $=1$, $z=$ Cord. Sup. of $\frac{1}{2m+1}$ Part of the Circumference, then $0=z^m-z^{m-1}-\frac{m-1}{1}z^{m-2}+\frac{m-2}{1}z^{m-3}+\frac{\overline{m-2}.\overline{m-3}}{2}z^{m-4}$ &c.

For if $\overline{2m+1}\times A=$ Semi-circumference, then the Cord of the Sup. and the general Series is $=0$, and may be divided by x. And in that Series $z=$ Cord. Sup. 2A, or of the $\overline{2m+1}^{\text{th}}$ Part of the Circumference.

PROP. XXXIII.

The Tangent of an Arch A *being given; to find the Sine and Coſine of n times the Arch.*

Let $t=$ Tan. A, then by Prop. XXIV. S.nA$=\frac{nx^{n-1}y}{r^{n-1}}-\frac{\overline{n-1}.\overline{n-2}.yy}{2.3.xx}$A &c. but $\frac{y}{x}=\frac{t}{r}$, and $x=\frac{rr}{\sqrt{rr+tt}}$; therefore $\frac{nx^{n-1}y}{r^{n-1}}$ or $\frac{nx^n ry}{r^n x}=\frac{nx^n t}{r^n}=\frac{nr^n t}{\overline{rr+tt}|^{\frac{n}{2}}}$, whence

$$\text{S.}n\text{A}=\frac{nr^n t}{\overline{rr+tt}|^{\frac{n}{2}}}-\frac{\overline{n-1}.\overline{n-2}.tt}{2.3rr}\text{A \&c.}$$

Like-

FIG.

Likewiſe by Prop. XXVI. Coſ. $nA = \frac{x^n}{r^{n-1}} - \frac{n.\overline{n-1}.yy}{2xx} A$ &c. that is $= \frac{r^{n+1}}{\overline{rr+tt}|^{\frac{n}{2}}} - \frac{n.\overline{n-1}.tt}{2rr} A$ &c. whence

$$S.nA = \frac{nr^n t}{\overline{rr+tt}|^{\frac{n}{2}}} - \frac{\overline{n-1}.\overline{n-2}.tt}{2.3rr} A - \frac{\overline{n-3}.\overline{n-4}.tt}{4.5rr} B - \frac{\overline{n-5}.\overline{n-6}.tt}{6.7rr} C - \frac{\overline{n-7}.\overline{n-8}.tt}{8.9rr} D - \text{\&c. and}$$

$$Coſ.\, nA = \frac{r^{n+1}}{\overline{rr+tt}|^{\frac{n}{2}}} - \frac{n.\overline{n-1}.tt}{2rr} A - \frac{\overline{n-2}.\overline{n-3}.tt}{3.4rr} B - \frac{\overline{n-4}.\overline{n-5}.tt}{5.6rr} C - \frac{\overline{n-6}.\overline{n-7}.tt}{7.8rr} D \text{ \&c.}$$

Cor. 1. Hence ſince $t = \frac{ry}{x}$, therefore *Tan.* $nA =$

$$\frac{nr^n t - \frac{n.\overline{n-1}.\overline{n-2}}{2.3} r^{n-2}t^3 + \frac{n.\dots.\overline{n-4}}{2.3.4.5} r^{n-4}t^5 \text{ \&c.}}{r^n - \frac{n.\overline{n-1}}{1.2} r^{n-2}tt + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4} r^{n-4}t^4 \text{ \&c.}}$$

Cor. 2. If $ſ =$ Secant of A, and $t =$ Tan. A, then *Secant* $nA =$

$$\frac{ſ^n}{r^{n-1} - \frac{n.\overline{n-1}}{2} r^{n-3}tt + \frac{n..\overline{n-3}}{1\dots4} r^{n-5}t^4 - \text{\&c.}}$$

for $\sqrt{\overline{rr+tt}} = ſ$, and $\frac{Rad.^2}{Coſine} =$ Secant.

SCHOLIUM.

Tho' the principal Uſe of the foregoing Propoſitions is when n is an integer Number, in which Caſe the Series will always terminate. Yet theſe Series will be equally true when n is any fractional Number, if they do but converge. For the Inveſtigation of any of theſe Propoſitions does not ſuppoſe that n is always an Integer, it only ſuppoſes that n is 1.

SECT.

SECT. V.

Angular Sections, the Inſcription of Polligons, the Properties of the Cords or Subtenſes of Arches.

PROP. XXXIV.

If any Arch of a Circle as AR *be given, and there be taken the Arch* $AD=\frac{1}{n}$ *the Arch* AR; *n being any integer Number; and if the whole Circumference of the Circle be divided into n equal Parts at the Points* D, E, F, G, H, *&c. beginning at* D: *Then I ſay any of the Arches* AD, AE, ADF, AEG, AEH, *&c. repeated n times, from* A, *will always end at* R. 7.

Call the whole Circumference C, then by Conſtruction, $n\times AD=AR$, alſo $n\times DE=C$, and $n\times DEF=2C$, and $n\times DEG=3C$, $n\times DFH=4C$, *&c.* therefore

$n\times AE$ or $n\times AD+n\times DE=AR+C$
and $n\times AF$ or $n\times AD+n\times DF=AR+2C$
and $n\times AEG$ or $n\times AD+n\times DEG=AR+3C$
$n\times AFH$ or $n\times AD+n\times EFH=AR+4C$, *&c.*

Therefore $n\times AE$, $n\times AF$, $n\times AEG$, *&c.* $=1, 2, 3$, *&c.* entire Circumferences + the Arch AR. And ſince any Number of Circumferences will end at the Point A where they began; conſequently by the Addition of the Arch AR, they will neceſſarily all end at R.

Cor. If the Cords AR, BR, AD, AE, AF, *&c.* are drawn; then the Cord AR= Cord of *n* times AD, or of *n* times AE, or of *n* times AEF, or of *n* times AEG, *&c.* and BR is the Cord of the Supplement of each of theſe.

PROP.

FIG.

PROP. XXXV.

7. *To divide an Arch* A R *into any Number of equal Parts as n.*

Let the Radius $=1$, $b=$ Cord of the Sup. of A R, or the Cord of B R, $x=$B D the Cord Sup. firſt Part A D. Then by Prop. XXX. $x^n - nx^{n-2} + \frac{n.\overline{n-3}}{2}x^{n-4}$ &c. $= \pm b$, according as the Arch A R is leſſer or greater than a Semi-circle. Therefore x the greateſt Root of this Equation, being extracted, will give BD the Cord of the Supplement of the Arch required.

EXAMPLE.

If the Arch A R be leſs than a Semi-circle, and is to be divided into 7 equal Parts; then $n=7$, and you will have $x^7 - 7x^5 + 14x^3 - 7x = b$, whoſe greateſt Root x is B D, the Cord of the Supplement of A D required.

Otherwiſe.

Let Rad. $=1$, Cord A R$=b$, Cord A D$=x$, then by Prop. XXIX. you will have $x^{2n} - 2nx^{2n-2} + 2n \times \frac{2n-3}{2}x^{2n-3}$, &c. to n terms $= \pm bb$, according as n is odd or even. Then extract the leaſt Root xx of this Equation, and then its Square Root x will be $=$ A D, the Cord of the firſt of the equal Parts. Or thus, put $nnxx - \frac{nn.\overline{nn-1}}{2.1.2.3}x^4 + \frac{nn.\overline{nn-1}.\overline{nn-4}}{3.1.2.3.4,5}x^6$ &c. $=bb$, &c.

EXAMPLE.

If A R was to be divided into 5 equal Parts, then $n=5$, and $x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 = bb$, whoſe leaſt Root xx is the Square of A D, and $x=$A D, the Cord of $\frac{1}{n}$ Part of the Arch A R.

Cor. If you continue to extract all the Roots x, you will get the Values of all the Cords B E, B F, B G, *&c.* by the former Method; or of the Cords A E, A F, A G, *&c.* by the latter Method; as appears by Cor. Prop. XXXII. FIG. 7.

SCHOLIUM.

If n is not a prime Number, but compounded of two or more Primes; it will be eaſier to divide the Arch into a Number of Parts denoted by one of theſe compounding Numbers, and then the firſt of theſe Parts into as many Parts as is denoted by another of them, and ſo on to the laſt.

PROP. XXXVI.

To inſcribe any regular Poligon in a Circle.

Let the Number of its Sides be $=2m+1$, an odd Number. Radius $=1$, $z=$ the unknown Cord B D, or the Cord of the Supplement of one Part A D. Then by Cor. Pr. XXXII. $z^m - z^{m-1} - \frac{m-1}{1} z^{m-2} + \frac{m-2}{1} z^{m-3}$, *&c.* $=0$, therefore the greateſt Root z will give B D, and $\sqrt{4-zz}=$ Side of the Poligon A D, required. 8.

EXAMPLE.

In a Heptagon, $2m+1=7$, or $m=3$, then by the former Equation $z^3 - z^2 - 2z + 1 = 0$, and the greateſt Root z gives the Value of B E, and $\sqrt{4-zz}=$A E the Side of the Heptagon. 9.

Otherwiſe.

Let $n=$ Number of Sides, Rad. $=1$. $x=$ Cord of the Sup. of the n^{th} Part of the Circle. Then ſince the Cord ſubtending the whole Circle is 0, therefore

FIG. by Prop. XXVIII. $yx^{n-1}-\frac{n-2}{1}yx^{n-3}$, &c. $=0$, therefore dividing by y, you will have $x^{n-1}-\frac{n-2}{1}x^{n-3}+\frac{\overline{n-3}\,.\,\overline{n-4}}{1\,.\,2}x^{n-5}$, &c. $=0$, extract the greateſt Root x of this Equation, then $\sqrt{4-xx}$=Side of the Poligon.

EXAMPLE.

In a Pentagon $n=5$; then $x^4-3x^2+1=0$, and x^2 = the greateſt Root, and $\sqrt{4-xx}$= Side of the Pentagon.

SCHOLIUM.

When the Number of Sides is not prime; it will be eaſier to divide the Circumference into a Number of Parts denoted by one of the Primes that compoſe it; and then one of theſe Parts into as many Parts as is denoted by another of the compounding prime Numbers, and ſo on till all the Primes which compoſe the given Number be exhauſted.

PROP. XXXVII.

10. *If the Arch* DAF *of a Circle, whoſe Diameter is* AE, *be divided into an even Number of equal Parts at* C, B, A, H, G, *&c. and there be drawn the lines* BH, CG, DF, *&c. perpendicular to the Diameter* AE; *and likewiſe the Cords* AB, BE. *I ſay, the Sum of the Lines,* BH, CG, *&c.* + *half the laſt, that is* $BH+CG+DO=\frac{AO\times EB}{AB}$.

For draw the Lines CH, DG, &c. then all the Triangles BAK, KHL, LCM, MGN, NDO, &c. are ſimilar. Therefore BK : KA : : HK : KL : : CM : ML : : GM : MN : : DO : ON, &c. therefore BK : AK : : BK+KH+CM+GM+DO : AK+KL+LM+MN+NO : : BH+CG+DO : AO.

$: OA$. But $EB : AB :: BK : AK :: BH+CG+DO : AO$, therefore $AB \times \overline{BH+CG+DO} = AO \times EB$. FIG. 10.

Cor. 1. If the Circle ADG be divided into an even Number of equal Parts at $A, B, C, D, E, F,$ *&c.* and all the Lines $BH, CG, DF,$ *&c.* be drawn, and alſo AB, BE. Then the Sum of all the Lines $BH+CG+DF,$ *&c.* $= \frac{AE \times EB}{AB}$.

For here the baſe of the Segment is 0, and the Points D, O, F, E coincide.

Cor. 2. If a Semicircle ABE be divided into any Number of equal Parts as n, at the Points $B, C, D,$ *&c.* and the Sines $BK, CM, DO,$ *&c.* be drawn; then I ſay, the Sum of all the Sines, $BK+CM+DO,$ *&c.* $= \frac{AE \times EK}{2BK}$.

For by ſimilar Triangles $\frac{EB}{AB} = \frac{EK}{BK}$; and taking half of the Quantities in Cor. 1. $BK+CM+DO = \frac{AE \times EB}{2AB} = \frac{AE \times EK}{2BK}$. Hence

Cor. 3. Let Radius $=r$, $\frac{180}{n}=A$, n being any Integer. $s=S.A$. $c=$ Coſ. A. then the Sum of all the Sines of A, 2A, 3A, 4A, *&c.* in the Semicircle, is $= \frac{r+c}{s}r$.

Cor. 4. Hence the Area of the Curve, made by erecting all the Sines perpendicular to a Line equal to the Semi-circumference, or as it is commonly called, the Area of the Figure of Sines, is $=2rr$.

For if n be infinite and A be $=1=s$, then $c=r$, and the Sum of all the Sines or Area of the Curve $= \frac{2rr}{s} = 2rr$.

FIG. 10.

PROP. XXXVIII.

If the Circumference of the Circle ADG *be divided into an even Number of equal Parts, at the Points* A, B, C, D, *&c. and the parallel Chords* AB, CH, DG, *&c. be drawn; I ſay, the Sum of all the Cords* $AB+CH+DG+EF$, *&c.* $=\frac{AE^2}{AB}$.

For by ſimilar Triangles $AE : AB :: AB : AK :: HL : KL :: CL : ML :: GN : MN :: DN : NO :: FE : EO$, and by Compoſition $AE : AB :: AB+CH+DG+FE : AE$.

Cor. 1. If a Circle is divided into an even Number of equal Parts, and A be one of the Parts. Then the Sum of all the Cords of 2A, 4A, 6A, 8A, *&c.* is to the Sum of all the Cords of A, 3A, 5A, 7A, *&c.* :: as EB : to EA.

For BH, CG, DF, *&c.* are the Cords of 2A, 4A, 6A. And BA, CH, DG, *&c.* are the Cords of 1A, 3A, 5A, *&c.* And the firſt Sum, is to the laſt as $\frac{AE\times EB}{AB}$ to $\frac{AE^2}{AB}$ or as EB : to AE.

Cor. 2. If a Semicircle is divided into any even Number of equal Parts, as n; and if A be one of theſe Parts, and $s=$S.A, Radius $=r$; then the Sum of the Sines of A, 3A, 5A, 7A, *&c.* $=\frac{rr}{s}$.

This follows from this Prop. taking half the Cord for the Sine of half the Arch.

Cor. 3. If a Semicircle is divided into an even Number of equal Parts, and A be one of theſe Parts, and Rad. $=r$, S.A$=s$, Coſ. A$=c$, then the Sum of all the even Sines in the Semicircle, that is the Sum of the Sines of 2A, 4A, 6A, 8A, *&c.* $=\frac{cr}{s}$.

For by Cor. 3. Pr. 37. Sum of the Sines of } A, 2A, 3A, 4A &c. $=\frac{r+c}{s}r$ FIG.

And by the laſt Cor. Sum of the Sines of } A, 3A, 5A &c. $=\frac{rr}{s}$

fore,
There Sum 2A 4A, 6A &c $=\frac{cr}{s}$

PROP. XXXIX.

If the Circumference of a Circle, whoſe Diameter is AB *and Center* C, *be divided into any odd Number of equal Parts, as n, at the Points* D, E, F, G, H, I, K; *and if, from any Point* A, *there be drawn the Cords* AD, AE, AF, AG, AH, *&c. and if there be taken the Arch* AR $= n \times$ *Arch* AD, *and the Cord* AR *drawn: I ſay the Product of all the Cords* AD×AE ×AF×AG×AH×AI×AK=AR×CB^{n-1}. 7.

1. Let the Line QAP be ſuppoſed to revolve about the Point A, the common Origin of all the Arches; then will the Part AP paſs thro' the Circumference of the Circle AEBHA. And if the Line ſtill continues to revolve about A, and the other End AQ will alſo paſs thro' all the Points of the Circumference AEBHA. And if the Line ſtill revolves forward, then the Part AP paſſes again thro' the Circle AEBHA, &c. and ſo on alternately. Now if from A to any Point of Interſection as R, the Cord AR be drawn, this Cord, AR will be affirmative in going the firſt time about the Circle; but negative, the ſecond time about; becauſe then AR will lie from A towards Q, in the Revolving line, which is the contrary way to what it laid before; likewiſe in going the third time about, AR will be affirmative, and the fourth time negative, and ſo on alternately. Therefore in general, let C= Circumference, A= any Arch leſs than C; then the Cords of A, 2C+A, 4C+A, &c. or of any even Number of C's +A, will be affirmative. And the Cords of

FIG. C+A, 3C+A, 5C+A, *&c.* or of any odd Number of C's +A, will be negative.

7. 2. Let A R$=c$, $x=$ Cord of the ſingle Arch. Now ſince n times the Arch A D, A E, A F *&c.* all end at R (by Prop. XXXIV.) and therefore have the ſame Cord A R, conſequently by Cor. 1. Pr. XXIX. we ſhall have

Caſe 1. $x^n - nx^{n-2}$ *&c.* $=+c$, the Cords of A, 2C +A, 4C+A *&c.*
or the Cords of $n\times$A D, $n\times$A F, $n\times$A H *&c.*
Caſe 2. $x^n - nx^{n-2}$ *&c.* $=-c$, the Cords of C+A, 3C+A, 5C+A *&c.*
or the Cords of $n\times$AE, $n\times$AG, $n\times$AEH *&c.*

And in Caſe 1$^{\text{ſt}}$ the Root A D, being affirmative, the others A F, A H *&c.* are affirmative for the ſame Reaſon.

And in Caſe 2$^{\text{d}}$ the Root A E is affirmative, and likewiſe the Roots A G, A I. But it is ſhewn in Algebra, that if the Signs of all the even Powers in an Equation be changed, the affirmative Roots are changed into negatives, and the contrary. And in Caſe 2$^{\text{d}}$ all the even Powers of x are wanting, except c, therefore in Caſe 2$^{\text{d}}$ changing $-c$ into $+c$, we have $x^n - nx^{n-2}$ *&c.* $=+c$, in which Equation, A E, A G, A I *&c.* are negative Roots. But this Equation is now the ſame as Caſe 1. therefore in general, in the Equation $x^n - nx^{n-2}$ *&c.* $=c$, the affirmative Roots are A D, A F, A H *&c.* and the negative ones A E, A G, A I *&c.*.

3. It is ſhewn in Algebra that in any Equation, as $x^n - nx^{n-2}$ *&c.* $=c$, or rather $x^n - nr^2x^{n-2}$ *&c.* $=cr^{n-1}$, the abſolute Number is equal to the Product of all the Roots; therefore A D$\times$A E$\times$A F$\times$A G *&c.* $=cr^{n-1}$.

Cor. 1. The Sum of all the odd Cords is equal to the Sum of all the even ones, A D+A F+A H+A K=A E+A G+A I.

For it is ſhewn in Algebra, that the Coefficient of the ſecond Term is = Sum of all the Roots; and here the ſecond Term is wanting, and therefore AD−AE+AF−AG &c. =0. FIG.

Cor. 2. If the Semi-circumference be divided into any odd Number of equal Parts, as *n*; whereof the two firſt are BF, the two next EF, the two next ED &c. and if the Cords AF, AE, AD &c. be drawn: Then the Product of all theſe Cords, AF×AE×AD=$CB^{\frac{n-1}{2}}$. 11.

For if AD=AK, then R and G fall in B, and AE=AI, AF=AH &c. therefore by this Prop. $AD^2\times AE^2\times AF^2\times AB=AB\times r^{n-1}$. 7.

Cor. 3. The ſame Things ſuppoſed as in the laſt Cor. I ſay, AF−AE+AD &c. =CB.

For by Cor. 1. 2AD+2AF=2AE+AB.

Cor. 4. If Arch BR=*n*×BG, and the Cord AR drawn, and the reſt be done as is ſuppoſed in the Prop. then the Product of all the Cords AD×AE×AF &c. =$AR\times CB^{n-1}$, and AD+AF+AH+AK=AE+AG+AI.

For it will appear with a little Conſideration, that AD+BG= half of DE=$\frac{C}{2n}$, or *n*×AD+*n*×BG=$\frac{1}{2}$C. But *n*×BG=BR, therefore *n*×AD=$\frac{1}{2}$C−BR=AR, therefore this Cor. comes to the Caſe of this Propoſition.

Cor. 5. If a Quadrant be divided into any odd Number of equal Parts, as *n*; and A be one of theſe Parts, Radius =*r*. then the Product of all the Sines, of 1 A, 3 A, 5 A, 7 A &c. in the Quadrant =

$$\overline{2\times\frac{r}{2}}\Big|^{\frac{n+1}{2}}.$$

This follows from Cor. 2. taking half the Cord for the Sine of half the Arch, and half Radius for CB, and multiplying both Sides by *r*.

FIG.

PROP. XL.

9. *If the Circumference of a Circle, whoſe Center is* C, *be divided into any odd Number of equal Parts as* n, *at the Points* A, E, F, G, H, I, K, *and from any one as* A, *to all the reſt there be drawn the Cords* A E, A F, A G *&c. I ſay the Product of all the Cords,* $AE \times AF \times AG \times AH \times AI \times AK = n \times AC^{n-1}$.

7. For let the Arches A D, A R be infinitely ſmall, and they will coincide with their Cords; then the Cord A D, is to the Cord A R, as 1 to n, and with this Ratio the Cords A D, A R vaniſh when A and D coincide, and therefore by Prop. XXXVII. $1 \times AE \times AF \times AG$ *&c.* $= n \times CB^{n-1}$.

Otherwiſe.

If A D be $=0$, then n times the Arch A E, A F, A G *&c.* always ends at A, and therefore c the Cord of theſe multiple Arches is $=0$, therefore if $s =$ Cord of the ſingle Arch, we ſhall have* (by Cor. 2. Prop. XXV.) $ns - \frac{n.nn-1}{24r^{2}}s^{3}$ *&c.* $=0$, wherein one Root s is $=0$, and the Equation being divided, we have $n - \frac{n^{3}-n}{24rr}s^{2}$ *&c.* $=0$, wherein the abſolute Number is n or rather nr^{n-1}, which therefore is equal to the Product of all the Roots, whoſe Number is now $n-1$.

9. *Cor.* 1. If the Semi-circumference A E B he divided into any odd Number of equal Parts, as n; whereof the two firſt is A E, the two next E F, the two next F G *&c.* And the Cords A E, A F, A G *&c.* be drawn: I ſay the Product of all the Cords $AE \times AF \times AG$ *&c.* $= CB^{\frac{n-1}{2}} \times \sqrt{n}$.

For $AE = AK$, $AF = AI$ *&c.* therefore $AE^{2} \times AF^{2} \times AG^{2} = n \times AC^{n-1}$.

Cor. 2. If the Quadrant of a Circle be divided into any odd Number of equal Parts as n, and A be one of them; then the Product of all the Coſines of A, 3A, 5A, 7A &c. $=\overline{\frac{r}{2}}\Big|^{\frac{n-1}{2}}\times\sqrt{n}$. FIG. 9.

This follows from the laſt Cor. becauſe $\frac{1}{2}$AG, $\frac{1}{2}$AF &c. are the Coſines of $\frac{1}{2}$BG, $\frac{1}{2}$BF &c. or of A, 3A &c.

Cor. 3. Hence alſo, the ſame things ſuppoſed as in this Prop. the ſum of the Squares of the Cords $AE^2+AF^2+AG^2+AH^2+AI^2+AK^2=2n\times CB^2$.

For in the Equation $ns-\frac{n^3-n}{24rr}s^3$ &c. $=0$, or rather by Cor. 1. Prop. XXIX. x^n-nx^{n-2} &c. $=0$, one Root $x=0$, which being divided, $x^{n-1}-nx^{n-3}$ &c. $=0$, ſince here the Index decreaſes by 2, therefore xx is the Root of the Equation, and n or nr^2 the Coefficient of the ſecond Term is the Sum of the Roots, that is of the Squares of AE, AF, AG; which doubled gives the Sum of all the Squares.

PROP. XLI.

If the Circumference of a Circle, whoſe Center is C, *and Diameter* AB, *be divided into any even Number of equal Parts, at the Points* D, E, F, G, H, I, *and from any Point* A, *there be drawn the Cords* AD, AE, AF *&c. And the Arch* AR *be taken* $=n\times$ *Arch* AD, *and the Cord* AR *drawn; then the Product of all the Cords* $AD\times AE\times AF\times AG\times AH\times AI=AR\times CB^{n-1}$. 12.

Let Cord $AR=c$, and $x=$ Cord of the ſingle Arch, then by Prop. XXIX. we have $-x^{2n}+2nx^{2n-2}$ &c. $=cc$. But xx or the Roots of this Equation are AD^2, AE^2, AF^2 &c. and therefore their Product $AD^2\times AE^2$ &6. $=cc$ or rather ccr^{2n-2}, and extracting the Root, $AD\times AE\times$ &c. $=cr^{n-1}$.

Cor.

FIG. *Cor*. 1. The Sum of the Squares of all the Cords $AD^2+AE^2+AF^2+AG^2+AH^2+AI^2=2n\times CB^2$.

For the Coefficient of the ſecond Term is the Sum of all the Roots $=2n$ or $2nr^2$.

13. *Cor*. 2. If the Semicircle AEB be divided into an even Number of equal Parts, as n; and the odd Cords AD, AE, AF drawn; I ſay their product $AD\times AE\times AF=CA^{\frac{n}{2}}\times\sqrt{2}$.

12. For in this Caſe, $AD=AI$, $AE=AH$, and $AF=AG$, and $AR=2CB$, therefore $AD^2\times AE^2\times AF^2=AR\times CB^{n-1}=2CB^n$.

PROP. XLII.

8. *If the Circumference of a Circle, whoſe Center is* C, *be divided into any even Number of equal Parts, at* A, D, E *&c. and from one of them as* A, *there be drawn to all the reſt the Cords* AD, AE, AB, AG, AH; I ſay the Product of all the Cords, $AD\times AE\times AB\times AG\times AH=n\times CB^{n-1}$.

For $n\times AD$, $n\times AE$ &c. all end at A, and their Cords $=0$, therefore by Cor. 1. Prop. XXIX. $x^{n-1}-\overline{n-2}.x^{n-3}$ &c. ..., $+\frac{1}{2}nx:\times\sqrt{4-xx},=0$, and one Root $x=0$, and dividing the Equation by it and the Correſpondent Cord of the Supplement $\sqrt{4-xx}$ (which here is AB), and we ſhall have $\pm x^{n-2}\mp\overline{n-2}\,x^{n-4}\ldots\ldots+\frac{1}{2}n=0$, where $\frac{1}{2}n$ or $\frac{1}{2}nr^{n-2}$ is the Rectangle of the remaining Roots AD, AE, AG, AH, which multiply by $2r$, and you have $AD\times AE\times AB$ &c. $=nr^{n-1}$.

This Prop. is alſo evident from the laſt Prop. by ſuppoſing AD, AR infinitely ſmall, and to vaniſh in the Ratio of 1 to n.

FIG.

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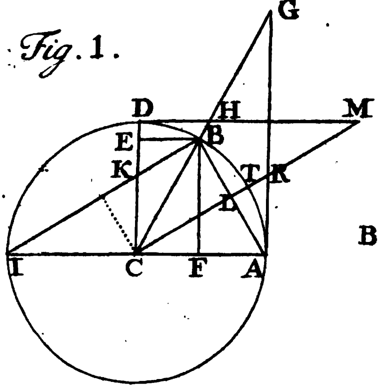

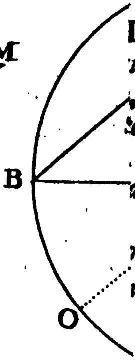

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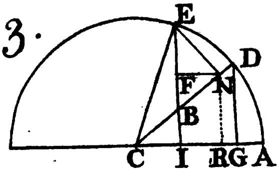

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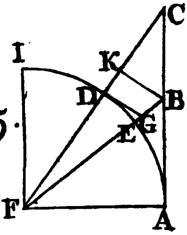

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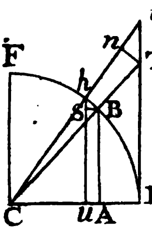

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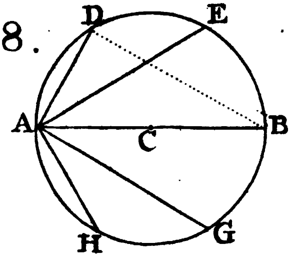

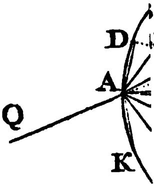

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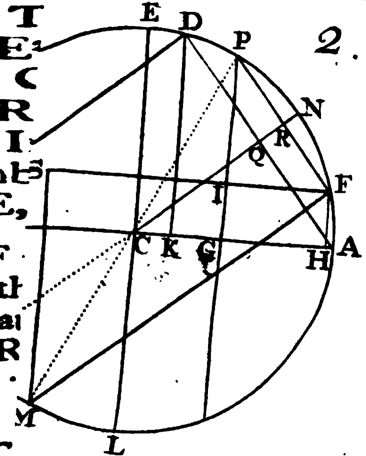

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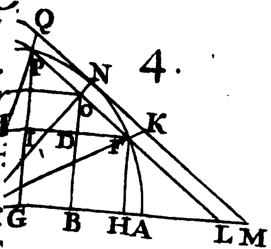

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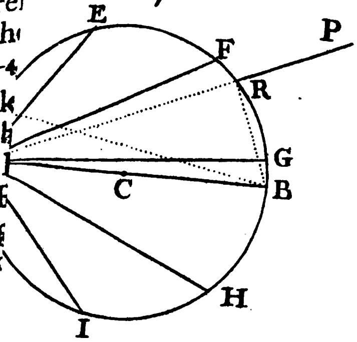

7.

Pl.I.*pa.*74.

FIG.

PROP. XLIII.

14. 15.

If the Circumference of a Circle AFHA, *whoſe Diameter is* AB *and Center* C, *be divided into any even Number of equal Parts as n, at the Points* D, E, F, G, H, I; *and from any Point* B *in the Circumference, there be drawn the Cords* BD, BE, BF *&c. and if the Arch* AR *be taken* $= n \times$ *Arch* AD, *and the Cord* BR *drawn; I ſay*

The Product of the Squares of the odd Cords $BD^2 \times BF^2 \times BH^2 = \overline{BA+BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is an odd Number; and* $= \overline{BA-BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is an even Number.*

And the Product of the Squares of the even Cords $BE^2 \times BG^2 \times BI^2 = \overline{BA-BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is odd, and* $= \overline{BA+BR} \times CB^{n-1}$, *when* $\frac{1}{2}n$ *is even.*

For Let A be the common Origin of the Arches; and let the line PQ revolve about B, beginning at A, and going thro' the Semi-circumference AEB in which any Cord as BR being drawn, will be affirmative. But after the Point of Interſection paſſes thro' B, the other Part BQ paſſes thro' the whole Circumference, in which any Cord BR will be negative, then the Part PB paſſes thro' the next Circumference from B, and any Cord drawn in it will be affirmative; and ſo on alternately. Therefore if C= Circumference, A= Arch AR leſs than $\frac{1}{2}$C; then will the Cords of A, 2C+A, 4C+A *&c.* be affirmative, and the Cords of C+A, 3C+A, 5C+A *&c.* negative. Put $a=$ Cord of BR, then

CASE I.

In the Arches AD, AF, AH, whoſe Sup. Cords are BD, BF, BH; we have $n \times AD = A$, $n \times AF = 2C+A$, $n \times AFH = 4C+A$ *&c.* wherefore the Cords of $n \times AD$, $n \times AF$, $n \times AFH$ are affirmative. Whence

FIG. Whence by Prop. XXX.

14. $x^n - nx^{n-2}$ &c. $-2 = +a$, when $\frac{1}{2}n$ is odd,

15. and $x^n - nx^{n-2}$ &c. $+2 = +a$, when $\frac{1}{2}n$ is even,

that is $x^n - nx^{n-2}$ &c. $= a \pm 2$, according as $\frac{1}{2}n$ is odd or even. In which Equation the Squares of BD, BF, BH the odd Cords are the Roots; therefore their Product is $2 \pm a$, or rather $\overline{2r \pm a} \times r^{n-1}$, according as $\frac{1}{2}n$ is odd or even.

CASE II.

In the Arches AE, AEG, AEI, whoſe Sup. Cords, are BE, BG, BI; we have $n \times AE = C + A$, $n \times AEG = 3C + A$, $n \times AEI = 5C + A$. Therefore the Cords of $n \times AE$, $n \times AEG$, $n \times AEI$, are negative, whence by Prop. XXX.

$x^n - nx^{n-2}$ &c. $-2 = -a$, when $\frac{1}{2}n$ is odd.
and $x^n - nx^{n-2}$ &c. $+2 = -a$, when $\frac{1}{2}n$ is even.

that is $x^n - nx^{n-2}$ &c. $+ a \mp 2 = 0$, as $\frac{1}{2}n$ is odd or even. Wherein the Squares of BE, BG, BI, the even Cords are the Roots, and therefore their Product is $a \mp 2$ or rather $\overline{2r \mp a} \times r^{n-1}$, according as $\frac{1}{2}n$ is odd or even.

14. *Cor.* 1. The ſame Things ſuppoſed; the Sum of the Squares of the odd Cords, as alſo the Sum of the Squares of the even Cords, $BD^2 + BF^2 + BH^2$ or $BE^2 + BG^2 + BI^2 = n \times CB^2$.

For the Coefficient of the ſecond Term in both Equations is n or nr^2, the Sum of the Roots.

Cor. 2. The Product of the Squares of the odd Cords + the Product of the Squares of the even Cords, that is, $BD^2 \times BF^2 \times BH^2 + BE^2 \times BG^2 \times BI^2 = 4CB^n$.

16. *Cor.* 3. Hence alſo if the Circumference be divided into an even Number of equal Parts as n; and Cords be drawn from any Point B to all the reſt, then the Sum of the Squares of the alternate Cords will be equal

qual to each other and to $n\times CB^2$; $BA^2+BF^2+BH^2+BL^2+BN^2=BE^2+BG^2+BM^2+BO^2=n\times CB^2$. FIG. 16.

This follows from Cor. 1. ſuppoſing the Point G to coincide with B, and D with A in Fig. 14, 15.

Cor. 4. If the Arch AR the multiple of AD, be greater than half the Circumference; then the Cord of the Supplement with be $-a$. In which Caſe you muſt write $-BR$ for $+BR$, and $+BR$ for $-BR$, in the preſent Propoſition.

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Since the Square of the Cord = verſed Sine × Diameter, therefore, in the foregoing Propoſitions, and their Corollaries, you may, by Subſtitution, find the Sums or Products of the verſed Sines of Arches, inſtead of the Sums or Products of the Squares of the Cords correſponding to them.

PROP. XLIV.

If the Circumference of a Circle, whoſe Diameter is AR, *and Center* O, *be divided into any Number of equal Parts as n, at the Points* A, C, E, G *&c. beginning at* A; *and from any Point* P *in the Diameter* AR, *be drawn the Lines* PC, PE, PG *&c. I ſay the Product of all the Lines* $AP\times PC\times PE\times PG$ *&c.* thro' the whole Circle is $=AO^n-OP^n$. 17 18

For draw the Cords AC, AE, AG; and from C, E *&c.* let fall Perpendiculars upon the Diameter AR, as CI *&c.* and let AO or OR $=r$, PO $=x$, AP $=y=r-x$. Then $PC^2=AP^2+AC^2-2PA\times AI=yy+AC^2-\frac{y}{r}AC^2$,

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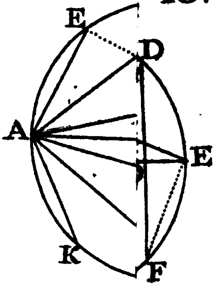
FIG. 17. 18. that is $PC^2=y^2+\frac{x}{r}AC^2$, likewiſe $PE^2=y^2+\frac{x}{r}AE^2$, alſo $PG^2=y^2+\frac{x}{r}AG^2$ &c. theſe multiply'd together produce $PC^2\times PE^2\times PG^2=y^6+\frac{x}{r}y^4\times\overline{AC^2+AE^2+AG^2}+\frac{xx}{rr}y^2\times\overline{AC^2\times AE^2+AC^2\times AG^2+AE^2\times AG^2}+\frac{x^3}{r^3}\times AC^2\times AE^2\times AG^2=PC\times Pc\times PE\times Pe\times PG\times Pg$. But in the ſecond Term of this Equation $AC^2+AE^2+AG^2$ is the Sum of the Squares of all the Cords (AC, AE, AG) which call A. And in the third Term is the Sum of all their Rectangles $=B$. In the fourth Term the Sum of all the Solids $=C$, and ſo on. Likewiſe the Index of y in the firſt Term is $n-1$, when n is odd, or $n-2$ when even. This being premiſed; the Equation will be expreſs'd in general thus, $PC\times Pc\times PE\times Pe$ &c. $=y^{n-1}+\frac{x}{r}y^{n-3}A+\frac{xx}{rr}y^{n-5}B+\frac{x^3}{r^4}y^{n-7}C+\frac{x^4}{r^4}y^{n-9}D$ &c. when n is odd. And $PC\times Pc\times PE\times Pe$ &c. $=y^{n-2}+\frac{x}{r}y^{n-4}A+\frac{xx}{rr}y^{n-6}B+\frac{x^3}{r^3}y^{n-8}C$ &c. when n is even. Let $x=$ Cord of AC, AE &c. then,

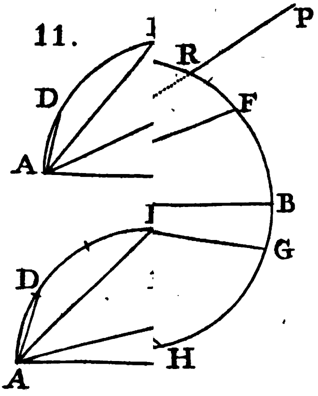
CASE I.

17. If n be an odd Number; then by Cor. 1. Prop. XXIX. the Cord of n times that Arch $=\frac{x^n}{r^{n-1}}-\frac{nx^{n-2}}{r^{n-3}}$ &c. $=0$ in this Caſe. But here one Root is $x=0$, by which dividing the Equation, and multiplying by r^{n-1},

10.



11.



14.

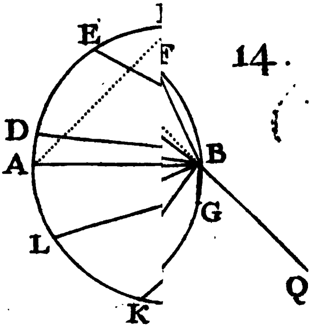


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FIG. 17. 18.

r^{n-1}, you will have $x^{n-1}-nr^2x^{n-3}+\frac{n.\overline{n-3}}{2}r^4x^{n-5}-\frac{n.\overline{n-4}.\overline{n-5}}{6}r^6x^{n-7}$ &c. $=0$, wherein the Roots are A C, A E, A G, A *g*, A *e*, A *c*. But ſince A C $=$ A *c*, A E $=$ A *e*, A G $=$ A *g*, and ſince in this Equation there are only the even Powers of x; therefore xx is the Algebraic Root of it, and all the particular Roots are AC^2, AE^2, AG^2.

But in every Equation the Coefficient of the ſecond Term is $=$ the Sum of all the Roots, therefore $AC^2+AE^2+AG^2=nr^2=A$, the Coefficient of the third Term $=$ Sum of all the Rectangles $=\frac{n.\overline{n-3}}{2}r^4=B$, the Coefficient of the fourth Term $=$ Sum of all the Solids $=C=\frac{n.\overline{n-4}.\overline{n-5}}{6}r^6$, and ſo on. Therefore putting for A, B, C &c. theſe their Values, and we ſhall have $PC\times PE\times PG\times Pg$ &c. $=$ $y^{n-1}+\frac{x}{r}y^{n-3}\times nr^2+\frac{x.x}{r.r}y^{n-5}\times\frac{n.\overline{n-3}}{2}r^4$ &c. $=y^{n-1}+nrxy^{n-3}+\frac{n.\overline{n-3}}{2}r^2x^2y^{n-5}+\frac{n.\overline{n-4}.\overline{n-5}}{6}r^3x^3y^{n-7}$ &c. that is, (putting $r-x$ for y and involving)

$$\begin{array}{lllll}
=r^{n-1}-\overline{n-1}.r^{n-2}x & +\frac{\overline{n-1}.\overline{n-2}}{2}r^{n-3}x^2 & & -\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{6}r^{n-4}x^3 & \text{\&c.} \\
\quad +nr^{n-2}x & -\frac{n.\overline{n-3}}{1}r^{n-3}x^2 & & +\frac{n.\overline{n-3}.\overline{n-4}}{2}r^{n-4}x^3 & \text{\&c.} \\
 & +\frac{n.\overline{n-3}}{2}r^{n-3}x^2 & & -\frac{n.\overline{n-3}.\overline{n-5}}{2}r^{n-4}x^3 & \text{\&c.} \\
 & & & +\frac{n.\overline{n-4}.\overline{n-5}}{6}r^{n-4}x^3 & \text{\&c.} \\
 & \text{\&c.} \ldots\ldots +x^{n-1} & & &
\end{array}$$

$$\begin{array}{lllll}
=r^{n-1}+r^{n-2}x & +\;r^{n-3}x^2 & + & r^{n-4}x^3 & \text{\&c.} \\
 & \text{\&c.} \ldots +x^{n-1} & & &
\end{array}$$

then

FIG. then multiply by PA or $r-x$, and you will have $PA\times PC\times PE\times PG\times Pg\times Pe\times Pc=r^n-x^n$.

CASE II.

18. If n be an even Number; then by Cor. 1. Prop. XXIX. the Cord of n times the Arch $=\sqrt{4rr-xx}\times\frac{x^{n-1}}{r^{n-1}}-\frac{\overline{n-2}\,.\,x^{n-3}}{r^{n-3}}$ &c. $=0$, in which is one Root $x=0$, and dividing the Equation by x and the correſpondent Cord of its Supplement $\sqrt{4rr-xx}$ (which here is AR), and multiplying by r^{n-1}, you will have $x^{n-2}-\frac{\overline{n-2}}{1}r^2x^{n-4}+\frac{\overline{n-3}\,.\,\overline{n-4}}{2}r^4x^{n-6}-\frac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6}r^6x^{n-8}$ &c. $=0$, in which there being only the even Powers of x, the Roots are AC^2, AE^2, AG^2.

But in this Equation the Coefficient of the ſecond Term is = Sum of all the Roots $AC^2+AE^2+AG^2=\overline{n-2}\times r^2=A$, the Coefficient of the 3^d Term is the Sum of all their Rectangles $=\frac{\overline{n-3}\,.\,\overline{n-4}}{2}r^4=B$. The Coefficient of the fourth Term is the Sum of all the Solids $=\frac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6}r^6=C$, and ſo on. Therefore putting for A, B, C &c. their Values in the general Equation, we have $PC\times PE\times PG\times Pg$ &c. $=y^{n-2}+\frac{x}{r}y^{n-4}\times\overline{n-2}\,.\,r^2+\frac{x^2}{r^2}y^{n-6}\times\frac{\overline{n-3}\,.\,\overline{n-4}}{2}r^4$ &c. $=y^{n-2}+\overline{n-2}\,.\,rxy^{n-4}+\frac{\overline{n-3}\,.\,\overline{n-4}}{2}r^2x^2y^{n-6}+\frac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6}r^3x^3y^{n-8}$ &c. that is (putting $r-x$ for y, and involving)

$=r^{n-2}$

$$\begin{array}{llllll}
=r^{n-2} & -\overline{n-2}\,.\,r^{n-3}x & +\dfrac{\overline{n-2}\,.\,\overline{n-3}}{2}\,r^{n-4}x^2 & -\dfrac{\overline{n-2}\,.\,\overline{n-3}\,.\,\overline{n-4}}{6}\,r^{n-5}x^3 & \\
 & +\overline{n-2}\,.\,r^{n-3}x & -\dfrac{\overline{n-2}\,.\,\overline{n-4}}{1}\,r^{n-4}x^2 & +\dfrac{\overline{n-2}\,.\,\overline{n-4}\,.\,\overline{n-5}}{2} & \&c. \\
 & & +\dfrac{\overline{n-3}\,.\,\overline{n-4}}{2}\,r^{n-4}x^2 & -\dfrac{\overline{n-3}\,.\,\overline{n-4}\,.\,\overline{n-6}}{2} & \&c. \\
 & & & +\dfrac{\overline{n-4}\,.\,\overline{n-5}\,.\,\overline{n-6}}{6} & \&c. \\
 & & & \&c. & \\
 & & \&c.\,.\,.\,.\,.\,.+x^{n-2} & & \\
=r^{n-2} & * & +r^{n-4}x^2 \quad * & +\quad r^{n-6}x^4 & \&c. \\
 & & \&c.\,.\,.\,.\,.+x^{n-2} & &
\end{array}$$

multiply by PA×PR or $rr-xx$, and you will have, $PA\times PC\times PE\times PG\times PR\times Pg\times Pe\times Pc=r^n-x^n$. *Q. E. D.*

Cor. 1. If the Semi-circumference AER be divided into any Number of equal Parts as n; at the Points A, B, C, D *&c.* and from any Point P in the Diameter AR be drawn Lines to all the even Points of Diviſion, as PC, PE, PG. I ſay, the Product of PA into that of all the Squares of theſe Lines, $PA\times PC^2\times PE^2\times PG^2$, if n is an odd Number, or of PA and PR into theſe Squares, $PA\times PR\times PC^2\times PE^2\times PG^2$ if n is even, is $=AO^n-OP^n$. FIG. 18.

For if $n\times AC=$ whole Circumference, then $n\times AB$ = half the Circumference, and $Pc=PC$, $Pe=PE$ *&c.*

Cor. 2. If a Semi-circle be divided into n equal Parts, and one of them be K, and s, t, u *&c.* be put for the Coſines of 2K, 4K, 6K *&c.* to $\overline{n-1}.K$, if n is odd, or to $\overline{n-2}.K$ if even; then I ſay $\overline{r-x}\times\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}$, *&c.* if n be an odd Number, or $\overline{r-x}\times\overline{r+x}\times\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}$ *&c.* (if n be even) $=r^n-x^n$.

FIG. 18. For let a, b, c &c. be the Sines of 2K, 4K, 6K &c. or of A C, A E, A G &c. then $CI=a$, $OI=s$, $PI=s-x$, $AP=r-x$, $PR=r+x$, and $PC^2=aa+\overline{s-x}^2=aa+ss-2sx+xx=rr-2sx+xx$, after the ſame manner $PE^2=rr-2tx+xx$, and $PG^2=rr-2ux+xx$, &c. and by Cor. 1. the Product of theſe ~~into $r-x$,~~ or elſe into $\overline{r-x}\times\overline{r+x}$, is $=r^n-x^n$. But obſerve that the Coſines of Arches greater than 90°, will be negative.

Cor. 3. Hence alſo if d, e, f &c. be the Sines of 1K, 2K, 3K, &c. then $y\times\overline{y^2+\frac{4ddx}{r}}\times\overline{y^2+\frac{4eex}{r}}\times\overline{y^2+\frac{4ffx}{r}}$ &c. if n be odd,

Or $y\times\overline{r+x}\times\overline{y^2+\frac{4ddx}{r}}\times\overline{y^2+\frac{4eex}{r}}\times\overline{y^2+\frac{4ffx}{r}}$ &c. if n be even, $=r^n-x^n$.

This follows from the Demonſtration of this Prop. for $2d=AC$, $2e=AE$, $2f=AG$, &c.

Cor. 4. Alſo if a, b, c be the verſed Sines of 2K, 4K, 6K &c.

$$\left.\begin{array}{l}\text{then if } n \text{ be odd, } y\\ \text{if } n \text{ be even } y\times\overline{r+x}\end{array}\right\}\times\overline{y^2+2ax}\times\overline{y^2+2bx}\times\overline{y^2+2cx},$$

&c. $=r^n-x^n$. For $a=\frac{2dd}{r}$, $b=\frac{2ee}{r}$, $c=\frac{2ff}{r}$.

17. 18. *Cor.* 5. And if P be taken without the Circle at π, the Product of all the Lines $\pi A\times\pi C\times\pi E\times\pi G$ &c. $=\pi O^n-AO^n$.

For let $O\pi : OA :: OA : OP$, and let the Lines PC, PE &c. be drawn; then ſince $\pi O : CO :: CO : OP$, and Angle O common, therefore the Triangles πOC and COP are ſimilar, and for the ſame Reaſon, πEO, and EOP, &c. Whence $\pi O : CO :: \pi C : PC :: \pi E : PE$ &c. $:: \pi R : PR :: \pi A : PA$. Therefore $PA=\frac{r\times\pi A}{\pi O}$, $PR=\frac{r\times\pi R}{\pi O}$, $PC=\frac{r\times\pi C}{\pi O}$, $PE=\frac{r\times\pi E}{\pi O}$ &c. $PO=\frac{rr}{\pi O}$. Whence $AP\times PC\times PE$

$\times PE\times PG \&c. = \frac{\pi A\times\pi C\times\pi E\times\pi G \&c.}{\pi O^n} r^n = r^n - \frac{r^{2n}}{\pi O^n}$, or $\pi A\times\pi C\times\pi E \&c. = \pi O^n - r^n$. FIG.

PROP. XLV.

If the Circumference of a Circle A E G, *whoſe Diameter is* A R, *and Center* O, *be divided into any Number of equal Parts as n, at the Points* B, D, F *&c. and the Part* B *b be biſſected by the Diameter* A R; *and from any Point* P *therein there be drawn the Lines* P B, P D, P F *&c. I ſay the Product of all theſe Lines* $PB\times PD\times PF$ *&c. thro' the whole Circle is* $= AO^n + OP^n$. 17. 18.

For let all the Parts B D, D F *&c.* be biſſected in C, E, *&c.* and P C, P E *&c.* drawn, and then the Circle will be divided into 2 *n* Parts, therefore

by the laſt Prop. } $PA\times PB\times PC\times PD\times PE \&c. = AO^{2n} - OP^{2n}$

and by the ſame, } $PA\times PC\times PE \&c. = AO^n - OP^n$

therefore by Diviſion } $PB\times PD\times PF\times PR\times Pf\times Pd\times Pb$ when *n* is odd,

or $PB\times PD\times PF\times PH\times Ph\times Pf\times Pd\times Pb$, when *n* is even, is $= \frac{AO^{2n} - OP^{2n}}{AO^n - OP^n} = AO^n + OP^n$.

Cor. 1. If the Semi-circumference, A E R be divided into any Number of equal Parts as *n*, in the Points A, B, C, D *&c.* and from any Point P in the Diameter A R, the Lines P B, P D, P F, *&c.* be drawn to all the odd Points of Diviſion: Then, the Product of all their Squares if *n* is even, $PB^2\times PD^2\times PF^2\times PH^2$; or that Product $\times d$ by P R that is $PB^2\times PD^2\times PF^2\times PR$ if *n* is odd, is $= AO^n + OP^n$.

For if $n\times Bb =$ whole Circumference, then $n\times AB = \frac{1}{2}$ Circumference, and $Pb = PB$, $Pd = PD$ *&c.*

FIG. 17. 18. *Cor.* 2. If a Semi-circle be divided into any Number of equal Parts as n, and one of them be K, and s, t, u *&c.* be the Coſines of 1K, 3K, 5K *&c.* to $\overline{n-1}.K$ when n is even, or to $\overline{n-2}.K$ when n is odd. Then I ſay

$\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}$ *&c.* $=r^n+x^n$, when n is even, or

$\overline{r+x}\times\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}$ *&c.* $=r^n+x^n$, when n is odd.

For if a, b, c *&c.* be the Sines of 1K, 3K, 5K *&c.* then the ſame way as in Cor. 2. Pr. XLIV. will be found $PB^2=rr-2sx+xx$, $PD^2=rr-2tx+xx$, $PF^2=rr-2ux+xx$ *&c.* whoſe Product alone, or elſe multiply'd by $r+x$, is, by Cor. 1. $=r^n+x^n$.

If P be without the Circle as at π, it will ſtill be $\pi B\times\pi D\times\pi F$ *&c.* $=AO^n+O\pi^n$.

For by this Prop. and Cor. 5. of the laſt; that Product will be $\frac{O\pi^{2n}-AO^{2n}}{O\pi^n-AO^n}=O\pi^n+AO^n$.

PROP. XLVI.

20. *If the Arch* Ae *of a Circle be divided into any Number of equal Parts as* n, *of which* AB *is one; and if the whole Circumference be divided into the ſame Number of equal Parts, at the Points* B, C, D, E *&c. beginning at* B; *and if from any Point* P, *taken in the Diameter* AR, *the Lines* PB, PC, PD *&c. be drawn, and* eF *be let fall perpendicular to* AR *to interſect it in* F, *between the Center* O *and* R. *And if* $AO=r$, $OP=$ $=x$, $OF=b$; *then I ſay the Product of all the Squares* $PB^2\times PC^2\times PD^2\times PE^2$ *&c.* $=r^{2n}+2br^{n-1}x^n+x^{2n}$.

For let $AO=r=1$. $AP=y=1-x$, then you will find as in Prop. XLIV. $PB^2=y^2+x\times AB^2$, $PC^2=y^2+x\times AC^2$, $PD^2=y^2+x\times AD^2$ *&c.* which multiply'd together, produce $PB^2\times PC^2\times PD^2$ *&c.* $=y^v+Axy^{v-2}+Bx^2y^{v-4}+Cx^3y^{v-6}$ *&c.* $+Fx^n$, putting $v=2n$, $A=$ Sum of the Squares of the Cords $AB^2+AC^2+AD^2$ *&c.*

&c. B= Sum of their Rectangles, C= Sum of their Solids *&c.* And F= Product of them all. FIG. 20.

Now if z be the Cord of an Arch, c the Cord of n times the Arch, then by Prop. XXIX. $z^{v} - vz^{v-2} + v \cdot \frac{v-3}{2} z^{v-4} - v \cdot \frac{\overline{v-4} \cdot \overline{v-5}}{2 \cdot 3} z^{v-6}$ *&c.* $= \pm cc$.

In which Equation, the Root zz repreſents indifferently the Square of any Cord A B, A C, A D *&c.* Therefore by common Algebra, the Sum of the Squares of the Cords, $AB^2 + AC^2 + AD^2$ *&c.* $= v$, the Sum of their Products $= v \cdot \frac{v-3}{2}$; Sum of their Solids $= v \times \frac{\overline{v-4} \cdot \overline{v-5}}{2 \cdot 3}$, *&c.* and the Product of all $= cc$. Whence we get $A = v$, $B = \frac{v \cdot \overline{v-3}}{2}$, $C = \frac{v \cdot \overline{v-4} \cdot \overline{v-5}}{2 \cdot 3}$ *&c.* and $F = cc$. Conſequently $PB^2 \times PC^2 \times PD^2$ *&c.* $= y^{v} + vxy^{v-2} + v \cdot \frac{v-3}{2} x^2 y^{v-4} + \frac{v\, \overline{v-4} \cdot \overline{v-5}}{2 \cdot 3} \times x^3 y^{v-6}$ *&c.* $+ ccx^{n}$, and putting $1 - x$ for y, and involving, we have $PB^2 \times PC^2 \times PD^2$ *&c.* $=$

$$\begin{array}{l} 1 - vx + v \cdot \frac{v-1}{2} x^2 - v \cdot \frac{v-1}{2} \cdot \frac{v-2}{3} x^3 \text{ \&c.} \ldots \text{ \&c.} - vx^{v-1} + x^{v} \\ \quad + vx - v \cdot \frac{v-2}{1} x^2 + v \cdot \frac{v-2}{1} \cdot \frac{v-3}{2} x^3 \text{ \&c.} \qquad \text{\&c.} + vx^{v-1} \\ \qquad + v \cdot \frac{v-3}{2} x^2 - v \cdot \frac{v-3}{2} \cdot \frac{v-4}{1} x^3 \text{ \&c.} \\ \qquad\qquad\qquad\qquad \text{\&c.} \\ \qquad\qquad\qquad\qquad + ccx^{n} \end{array}$$

Hence it will appear, by ſuming them up, that the Coefficients of the ſecond, third *&c.* Terms are o, till you come to the middle Term, where you have ccx^{n} (the Index n being $\frac{1}{2}v$); likewiſe beyond the middle Term they are o, becauſe the Coefficients are

the

FIG. 20. the ſame at equal Diſtances on each Side from the Middle; whence they all vaniſh till the laſt x^v.

Now to find the Coefficient of the middle Term, it muſt be obſerved that ccx^n ſupplies the Place of the firſt Term of the next horizontal Series; which Term, if it were there, then the Sum of the Coefficients (in that perpendicular Column) would there alſo be $=0$, as before. Hence it follows that if $v=2$, 4, 6 &c. that the Term vx, $v.\frac{v-3}{2}x^2$, $v.\frac{v-4}{2}.\frac{v-5}{3}x^3$ &c. will be wanting, reſpectively, which Term is $2x$, $2xx$, $2x^3$, &c. reſpectively, and in general the Term wanting is always $\frac{1.2.3\ldots\overline{n-1}\times v}{1.2.3\ldots\overline{n-1}.n}x^n=\frac{v}{n}x^n$ $=2x^n$. Conſequently the middle Term will be found to be $-2x^n+ccx^n$. But $cc=AR\times AF=2+2b$, therefore $-2x^n+\overline{2+2b}.x^n=2bx^n$ is the middle Term. And therefore $PB^2\times PC^2\times PD^2$ &c. thro' the whole Circle, $=1+2bx^n+x^{2n}$, or rather $r^{2n}+2r^{n-1}bx^n+x^{2n}$.

21. *Cor.* 1. If the Arch A *e* be leſs than a Quadrant, and F lye from O towards A; then will $PB^2\times PC^2\times PD^2\times PE^2$ &c $=r^{2n}-2br^{n-1}x^n+x^{2n}$.

This follows from this Prop. putting $-b$ inſtead of $+b$, as lying on the contrary Side of O.

20. 21. *Cor.* 2. If P be without the Circle as at π, then will $\pi B^2\times\pi C^2\times\pi D^2$ &c. $=r^{2n}\pm 2br^{n-1}x^n+x^{2n}$, according as F falls beyond, or on this Side of O, in reſpect to P.

The Demonſtration of this Prop. equally proves this, ſince the even Powers of $x-1$, are the ſame with thoſe of $1-x$, and v or $2n$ is an even Number.

21. *Cor.* 3. Let $b=$ Coſine of an Arch A or A *e*, to the Radius r; and $\frac{A}{n}=B$, $\frac{360}{n}=C$, and s, t, u &c. the Coſines of B, B+C, B+2C, B+3C, &c. to $B+\overline{n-1}.C$,

$\overline{n-1}$. C, to the Radius r, $x=$ any Quantity: Then I ſay, $\overline{rr-2sx+xx}\times\overline{rr-2tx+xx}\times\overline{rr-2ux+xx}$ &c. to n Terms, is $=r^{2n}-2br^{n-1}x^{n}+x^{2n}$. FIG. 21.

For let $a=$BI the Sine of B, $s=$OI the Coſine, then $PB^2=BI^2+PI^2=aa+\overline{s-x}^2=aa+ss-2sx+xx$ $=rr-2sx+xx$, for the ſame Reaſon, $PC^2=rr-2tx$ $+xx$, $PD^2=rr-2ux+xx$, &c. but by Cor. 1. the Product of theſe is $=r^{2n}-2br^{n-1}x^{n}+x^{2n}$.

If A be greater than a Quadrant, write $+b$ for $-b$. Likewiſe any of the Coſines s, t, u &c. of Arches between 90, and 270, is negative, and therefore in that Caſe write $+s$, $+t$, $+u$ inſtead of $-s$, $-t$, $-u$, &c.

THE ELEMENTS OF TRIGONOMETRY.

BOOK II.

Plain Trigonometry, or the Doctrine of plain Triangles.

PLAIN *Trigonometry* teaches the Relation of the Sides and Angles of Plain Triangles, and how to calculate their Sides and Angles.

A Plain Triangle is made by three right Lines, which are called it's Sides, and if one Side be perpendicular to another, or makes a right Angle with it; then 'tis call'd *a right Angle-Triangle*, otherwiſe 'tis an oblique Triangle.

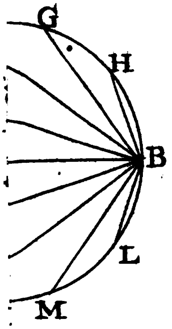
A right Angle is meaſured by an Arch of 90°, an *acute Angle* is leſs, and an *obtuſe* one greater than a right Angle.

In a right Angled-triangle, the Side oppoſite to the right Angle is call'd the *Hypothenuſe* ; and the other two the *Legs* or *Sides*, or ſometimes the *Perpendicular and Baſe.*

In all plain Triangles the Sum of the three Angles together is 180 Degrees: And in a right Angle-triangle, the Sum of the two oblique Angles is 90°.

When three Letters denote an Angle, the middle Letter is always at the Angle.

SECT.



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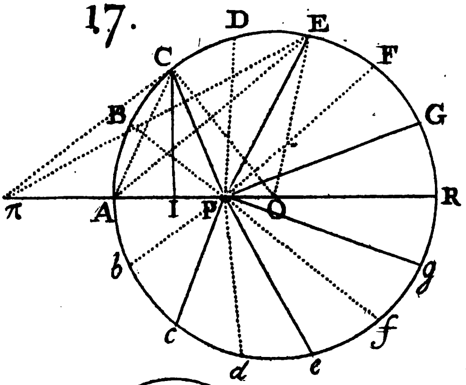

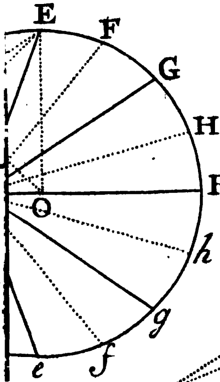

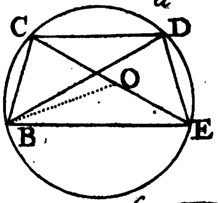

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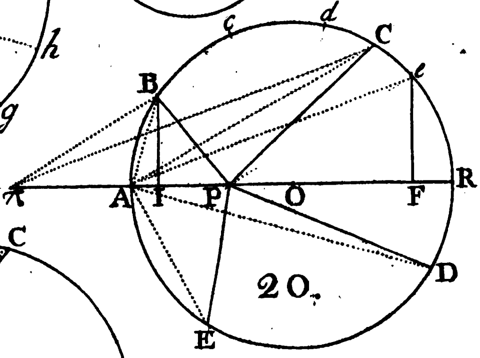

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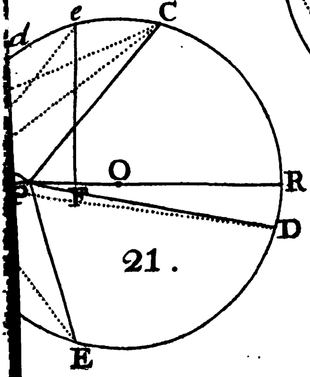

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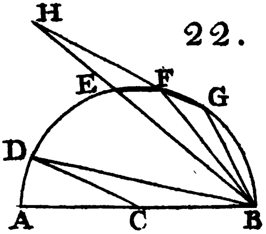

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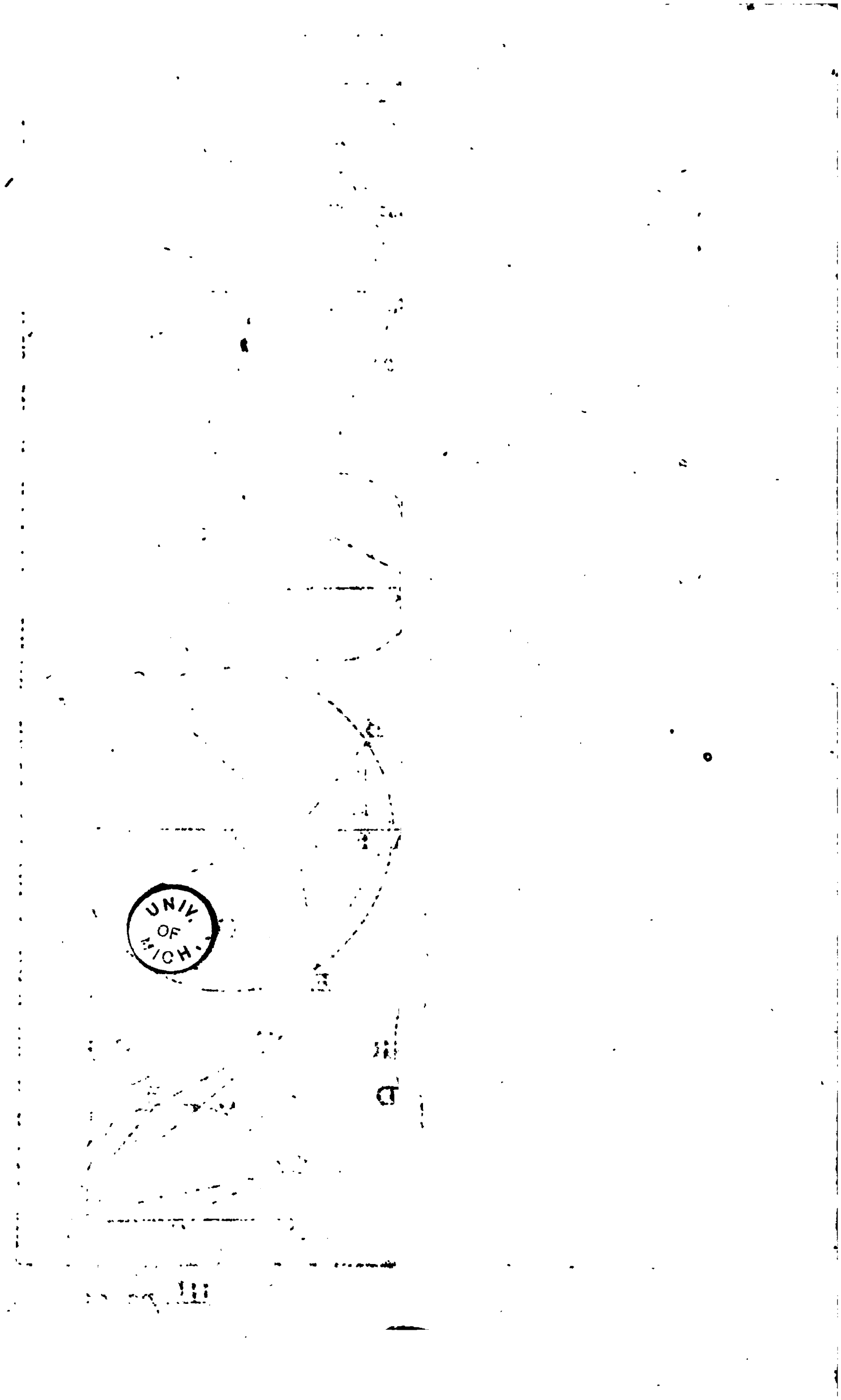

UNIV. OF MICH.

SECT. I.

The Relations, and ſeveral Proportions among the Sides, and Angles of plain Triangles.

PROP. I.

In any right angled Triangle C A B, *let the Hypothenuſe* $CA = b$, *and* $CA - CB$ *or* $BD = v$; *then,* FIG. I. $57{\cdot}29578\sqrt{\frac{2v}{b}} + \frac{v}{3{\cdot}4b}A + \frac{3^2v}{5{\cdot}8b}B + \frac{5^2v}{7{\cdot}12b}C$, *&c.* = *Angle* A C B *in Degrees*; *where* A, B, C *are the preceding Terms.*

For with the Center C deſcribe the Arch A D, then B D or v is the verſed Sine of the Arch A D, and Radius b, therefore (by Prop. XIV. B. I) Arch $AD = \sqrt{2bv} + \frac{v}{3{\cdot}4b}A + \frac{3^2v}{5{\cdot}8b}B + \frac{5^2v}{7{\cdot}12b}C$ *&c.* but as half the Circumference $3{\cdot}14159b$: 180 Degrees : : A D : $\frac{180\,AD}{3{\cdot}14159}$ = Degrees in the Arch $AD = \frac{57{\cdot}295779}{b} \times AD$, that is, $\angle ACD = \frac{57{\cdot}29}{b} \times 2bv + \frac{v}{3{\cdot}4b}A$: *&c.* $= 57{\cdot}2957\sqrt{\frac{2v}{b}} + \frac{v}{3{\cdot}4b}A + \frac{3^2v}{5{\cdot}8b}B$ *&c.*

Cor. 1. *As the Sum of the Hyp. and* $\frac{1}{2}$ *longer Leg,*
$AC + \frac{1}{2}CB$:
To 86 : :
So the ſhorter Leg, A B :
To its oppoſite Angle C, *nearly.*

Let $CB = b$, and $AB = c$, $a = \angle C$ in Degrees, then by the Nature of the Circle $c = \sqrt{2bv - vv} = \sqrt{2bv} - \frac{v}{4b}\sqrt{2bv}$ *&c.* by Evolution. Then by this Prop. $a =$

$\frac{57\cdot3}{b}\times:\sqrt{2bv}+\frac{v}{12b}\sqrt{2bv}$: nearly, multiply this Equation by the former, and you have $a\times:\sqrt{2bv}-\frac{v}{4b}\sqrt{2bv}=\frac{57\cdot3c}{b}\times:\sqrt{2bv}+\frac{v}{12b}\sqrt{2bv}$, and dividing by $\sqrt{2bv}$, $a\times\overline{1-\frac{v}{4b}}=\frac{57\cdot3c}{b}\times\overline{1+\frac{v}{12b}}$. And $\frac{57\cdot3c}{b}$

$$=a\times\frac{1-\frac{v}{4b}}{1+\frac{v}{12b}}=\text{(by Divifion) } a\times\overline{1-\frac{v}{3b}\ \&c.}$$

$=a\times\frac{3b-v}{3b}=a\times\frac{3b-b+b}{3b}=a\times\frac{2b+b}{3b}$, whence $a\times\overline{2b+b}=171.9c=172c$ nearly, or $\overline{b+\frac{1}{2}b}\times a=86c$.

SCHOLIUM.

If $c=1\,.\,n=57\cdot3$, then $\frac{3}{2}n=86$, and by Cor. 1. $b+\frac{1}{2}b:1::\frac{3}{2}n:a$, and $2b+b=\frac{3n}{a}$, and $b=\frac{3n}{2a}-\frac{b}{2}$ $=\frac{3n}{2a}-\frac{\sqrt{bb-1}}{2}$, whence $\frac{3n}{a}-2b=\sqrt{bb-1}$. this Equation reduced gives, $b=\frac{2n}{a}-\sqrt{\frac{nn}{aa}-\frac{1}{3}}$. Whence $b=\frac{3n}{a}-2b$; which is *Forſter*'s Rule.

PROP. II.

1. *If any right angled Triangle* CAB, *let* $a=$ *Degrees in the leſſer* ∠ C, $m=\frac{a}{57\cdot29578}$. *Then*

As 1 :

To $1-\frac{m^2}{1.2}-\frac{m^2}{3.4}A-\frac{m^2}{5.6}C-\frac{m^2}{7.8}D$ &c. ::

As the Hypothenuſe, C A :
: *To the adjoining Side* C B.

where A, B, C *&c.* are the foregoing Terms, with their Signs.

For let Radius C A or C D $=b$. Arch A D $=z$, then $z=\frac{ab}{57.295\ \&c.}=mb$, and by Cor. 1. Prop. XII. Coſine $CB=b-\frac{z^2}{1.2b}+\frac{z^4}{1.2.3.4b^3}\ \&c.\ =b-\frac{m^2b^2}{1.2b}+\frac{m^4b^4}{1.2.3.4b^3}-\&c.=CA\times\overline{1-\frac{m^2}{1.2}+\frac{m^4}{1.2.3.4}-\frac{m^6}{1\ldots6}+\frac{m^8}{1\ldots8}\ \&c.}$

Cor. 1. *If a be the leſſer Angle, in Degrees, then it will be,*

As $\frac{57.3}{a}+\frac{3a}{1000}:1::$ *Hyp.* C A : *op. ſide* A B, *nearly*.

For let $n=57.3$, then $CB^2=bb-z^2+\frac{z^4}{3b^2}\ \&c.$ and $AB^2=CA^2-CB^2=z^2-\frac{z^4}{3b^2}$, and extracting the Root, $AB=z-\frac{z^3}{6bb}=mb-\frac{m^3b}{6}=b\times\overline{\frac{a}{n}-\frac{a^3}{6n^3}}$; and dividing both Sides by $\frac{a}{n}-\frac{a^3}{6n^3}$, $b=AB\times\overline{\frac{n}{a}+\frac{a}{6n}}=$ $:\frac{57.3}{a}+\frac{3a}{1031}\times:AB.$

Cor. 2. *If a be the leſſer Angle in Degrees, then*

As $1:1-\frac{1\frac{1}{2}aa}{10000}::$ Hyp. C A : adjoining Side B C, nearly.

For $CB=CA\times\overline{1-\frac{m^2}{2}}=CA\times\overline{1-\frac{aa}{2nn}}=CA\times\overline{1-\frac{1.52aa}{10000}}$.

SCHO-

FIG.

SCHOLIUM.

1. Let $q=90°$, $y=q-a$, then ſince $CB=CA\times\overline{1-\frac{aa}{2nn}}$, by Diviſion $CA=CB\times\overline{1+\frac{aa}{2nn}}$, and $CA\times y=CB\times y+\overline{\frac{4aa}{\frac{8}{y}nn}}$. But $\frac{8nn}{y}=\frac{8nn}{q-a}=\frac{8nn}{q}+\frac{8nna}{qq}$ &c. $=292+3a$. Therefore $CA\times y=CB\times y+\overline{\frac{4aa}{300+3a}}$, nearly, which is *Wilſon*'s Rule.

Since in a right Angled-triangle, the Square of the Hypothenuſe is equal to the Sum of the Squares of the Sides. And in any Triangle, the Baſe is to the Sum of the Sides, as their Difference to the Difference of the Segments made by a Perpendicular. Therefore by theſe and the two foregoing Propoſitions and their Corollaries, all the Caſes of plain Triangles may be reſolved, without any Tables whatever: Dividing oblique Triangles into two right angled ones, by a Perpendicular let fall from the End of a given Side and oppoſite to a given Angle.

PROP. III.

1. *In any right angled Triangle* C A B, *any one of the three Sides being made a Radius; and a Circle deſcribed from the End of it as a Center; then the other Sides will repreſent Sines, Tangents, Secants, &c. of the reſpective Angles: Then it will be as any Side; to what it repreſents in the Table :: ſo is any other Side, to what it repreſents in the Table.*

Deſcribe the Arches A D and B E, from the Center C, then if C A be made Radius; A B will be the Sine, and C B the Coſine, of the Angle C. And if C B be made Radius, A B will be the Tangent, and C A the Secant of the Angle C. And becauſe the

Sines, Tangents &c. of any Angles may be found in the Tables, therefore when any Triangle is propoſed, there may always be found a ſimilar one in the Tables; and conſequently the Sides about the equal Angles are proportional. Therefore it will be, FIG. 1.

As Radius : AC : : S . C : A B :: Coſ. C : C B,
and Rad. : C B : : Tan. C : A B :: Sec. C : C A.

PROP. IV.

In any right angled Triangle C D E, 2.
As the Sum of the Hyp. and one Side, CE+ED :
Radius : :
So the other Side, C D :
To Tan. half the op. Angle C E D.

Produce DE to B, ſo that EB=EC, then ∠B =∠BCE, and Angle DEC=∠B+BEC=2∠B. But in the Triangle CDB, by Prop. III. it is as DB : Radius : : CD : Tan. B, that is DE+EC : Rad. : : CD : Tan. $\frac{1}{2}$E.

PROP. V.

The Sides of any Triangle, are proportional to the Sines of their oppoſite Angles, AB : S.F : : AF : S.B : : FB : S.A. 3.

On AB, AF let fall the Perpendiculars FH, B*b*;
then by Prop. III. AF : FH : : R : S.A
FH : FB : : SB : R
therefore ex equo, AF : FB : : S.B : S.A.

Again,

As AB : B*b* : : R : SA
and B*b* : BF : : S.F : R
and ex equo AB : BF : : S.F : S.A.

Cor. 1. If a Line AD be drawn from the Vertex of any Triangle cutting the Baſe; the Rectangle of the 4.

FIG. 4. the Sides and Sines of the vertical Angles, are directly as the Segments of the Baſe.

$$\text{For } S.CDA : CA :: S.CAD : CD = \frac{CA \times S.CAD}{S.CDA}.$$

And $S.CDA$ or $BDA : AB :: S.DAB : DB = \frac{AB \times S.DAB}{S.CDA}$ therefore $CD : DB :: CA \times S.CAD : AB \times S.DAB$.

5. *Cor.* 2. If in a Triangle two Sides A C, C B, and the Angle A oppoſite to the leſſer Side C B, be given; the Angle oppoſite to the greater may be either leſs or greater than a right Angle; alſo the Baſe and it's oppoſite Angle, will have each of them, two ſeparate Values. Therefore the three Things required are ambiguous.

For make $CD = CB$, then $\angle CDB = \angle B$, and there are the two Triangles A C B, A C D, wherein two Sides and an oppoſite Angle are given the ſame in both; therefore the other three things may be found in either Triangle; for the Angle oppoſite to A C may be either B or it's Sup. A D C, and the Baſe may be either A B or A D, and it's op. Angle may be either A C B or A C D.

PROP. VI.

6. *In any Triangle* A B C,
As the Sum of any two Sides $BC + AB$:
To their Difference $BC - AB$ ::
Tan. half the Sum of their op. Angles, Tan. $\frac{A+C}{2}$:
Tan. half their Difference, Tan. $\frac{A-C}{2}$.

Make $BD = BA$, and draw A D, and B F G $\perp$ to it; likewiſe F E $\parallel$ A C, then will $AF = FD$, $DE = EC$.

EC. The $\angle BAF = \frac{BAD+BDA}{2} = \frac{BAC+C}{2}$, FIG. 6.
and $\angle GAF = BAC - BAF = \frac{BAC-C}{2}$.

The Triangles BFE, BGC are ſimilar, whence BE : EC :: BF : FG. But AF : R :: BF : Tan. BAF :: FG : Tan. FAG, or BF : FG :: Tan. BAF : Tan. FAG. Conſequently BE or $\frac{BD+BC}{2}$: EC or $\frac{BC-BD}{2}$ :: Tan. BAF : Tan. FAG.

Otherwiſe.

BC : BA :: S.A : S.C. and BC+BA : BC−BA :: S.A+S.C : S.A−S.C :: (by Cor. 1. Pr. VII.) Tan. $\frac{A+C}{2}$: Tan. $\frac{A-C}{2}$.

Note. Inſtead of the Tan. half the Sum of the op. Angles, you may take the Cotangent of half the included Angle, which is equal to it.

Cor. 1. As the Baſe CA :
Sum of the Sides, CB+BA ::
So diff. Sides CD :
Diff. Segments of the Baſe made by a Perpendicular ::
So Coſ. ½ Sum op. Angles, or S. ½ the vertical Angle :
Coſ. ½ diff. op. Angles, or Coſ. ½ diff. vertical Angles, made by a Perp.

For ſince $DF = \frac{1}{2}DA$, $FE = \frac{1}{2}AC$; and in the Triangle FBE, FE : BE :: S.FBD or Coſ. BDF : S.BFE or GFE or Coſ. DFE. But DFE or $DAC = BDA - C = BAD - C$, and $2\,DAC = BAC - C$, alſo $\frac{1}{2}B =$ Comp. $\frac{A+C}{2}$, and $A - C =$ diff. vert. ∠S made by a Perpendicular.

Likewiſe

FIG. 6. Likewiſe it is evident from Geometry, that $CA : CB+BA :: CD$: diff. Segments of the Baſe. Suppoſing a Circle deſcribed with Center B and Radius B A. See *Eucl.* III. Prop. XXXVI. and Cor.

Cor. 2. As the Baſe C A :
Difference of the Sides C D : :
So Sum of the Sides $CB+BA$:
Diff. Segments by a Perpendicular :
So Sine half Sum op. Angles, $\frac{A+C}{2}$:
Sine of half their Difference, $\frac{A-C}{2}$.

For in the Triangle A D C, A C : D C : : S.ADB : S.C A D.

Cor. 3. As Square of the Baſe AC^2 :
Difference of the Squares of the Sides, $BC^2 - BA^2$: :
S. Sum op. Angles, $A+C$:
S. their difference, $A-C$.

For by multiplying the two Proportions in Cor. 2 and 1. $AC^2 : \overline{CB+BA} \times \overline{CB-BA} ::$ Coſ. $\frac{A+C}{2} \times$ S. $\frac{A+C}{2}$: Coſ. $\frac{A-C}{2} \times$ S. $\frac{A-C}{2}$: : (by Cor. 3. Pr. B.I. II.) S.$\overline{A+C}$: S.$\overline{A-C}$.

PROP. VII.

7. *In any Triangle* A F B,
As twice the Rectangle of the Legs, $2\,AF \times FB$:
Sum of Squares of the Legs — Square Baſe : :
Radius :
Coſ. vertical Angle, F.

On F B let fall the Perpendicular A C, then by Prop. XIII. *Eucl.* II. $AB^2 + 2\,BF \times CF = AF^2 + FB^2$, and $2BF \times FC = AF^2 + FB^2 - AB^2$. But by Prop. III. A F : F C : : Rad : Coſ. F. And multiplying the two firſt Terms by $2FB$, then will $2AF\times FB : 2BF\times FC$ or $AF^2+FB^2-AB^2 :: \text{Rad} : \text{Coſ. } F.$ FIG. 7.

Cor. 1. Hence theſe are reſpectively proportional,
As twice the Rectangle of the Legs :
Rectangle of the Baſe + diff. Legs, and Baſe — diff. Legs :
Rectangle of the Sum of the three Sides, and Sum Legs — Baſe : :
So Radius :
Verſed Sine of the included Angle :
Verſed Sine of the Sum of the oppoſite Angles.

For let $m=AF$, $n=BF$, $b=AB$, then by this Prop.

$r : \text{Coſ. } F :: 2mn : mm+nn-bb$; and by Compoſition,

$r : r\pm\text{Coſ. } F :: 2mn : 2mn\pm mm\pm nn\mp bb$,

that is by Schol. Prop. I.

$$r : \left\{\begin{array}{l}\text{verſ. Sup. } F\\ \text{verſ. } F\end{array}\right\} :: 2mn : \left\{\begin{array}{l}\overline{m+n}^2-bb=\overline{m+n+b}\times\overline{m+n-b}\\ bb-\overline{m-n}^2=\overline{b+m-n}\times b-\overline{m-n}.\end{array}\right.$$

Cor. 2. From half the Sum of the three Sides, ſubtract each Side ſeverally, and note the three Remainders. Then
As half the Rectangle of the Legs :
To Square Root of the Product of this half Sum and the three Remainders : :
So Radius :
Sine of the vertical Angle.

For let $s=b+m+n$, $v=$ verſ. F. $V=$ verſ. Sup. F. Then by Cor. 1.

$v=\frac{\overline{b+m-n}\times\overline{b+n-m}}{2mn}r=\frac{\overline{s-2n}\times\overline{s-2m}}{2mn}r$, and $V=\frac{\overline{b+m+n}\times\overline{m+n-b}}{2mn}r=\frac{s\times\overline{s-2b}}{2mn}r$. And by Schol. Prop. I.

FIG. 7. $\sqrt{Vv} = S.F = \frac{r}{2mn}\sqrt{s \times \overline{s-2b} \times \overline{s-2m} \times \overline{s-2n}} = \frac{2r}{mn} \times$
$\sqrt{\frac{s}{2} \times \frac{s-2b}{2} \times \frac{s-2m}{2} \times \frac{s-2n}{2}}.$

Cor. 3. As the Rectangle of the Legs :
Radius Square : :
So the Rectangle $\frac{1}{2}$ Sum 3 Sides — one Leg, and that $\frac{1}{2}$ Sum — other Leg :
Sine Square of half the vertical Angle : :
And ſo the Rectangle $\frac{1}{2}$ Sum 3 Sides, and that $\frac{1}{2}$ Sum — Baſe :
Coſine Square of half the vertical Angle.

For multiplying the three laſt Terms in Cor. 1. by $\frac{1}{2}r$; then $2mn : \overline{b+m-n} \times \overline{b+n-m} : s \times \overline{s-b} :: \frac{1}{2}rr : \frac{1}{2}rv : \frac{1}{2}rV$. But by Prop. II. $\frac{1}{2}rv =$ Sine Square of half F, and $\frac{1}{2}rV =$ Coſ. Square of $\frac{1}{2}$F. Therefore $2mn : \overline{s-2m} \times \overline{s-2n} : s \times \overline{s-b} :: \frac{1}{2}rr :$ Sine^2 : Coſine^2, or $mn : \frac{s-2m}{2} \times \frac{s-2n}{2} : \frac{s}{2} \times \frac{s-b}{2} :: rr :$ Sine^2 : Coſine^2, of $\frac{1}{2}$F. And hence alſo follows.

Cor. 4. As four Times the Rectangle of the Legs :
To Radius Square : :
So Rectangle of the Baſe + diff. Legs, and Baſe — diff. Legs :
Sine Square of half the vertical Angle : :
And ſo Rectangle of Sum Legs + Baſe, and Sum Legs — Baſe :
Coſine Square of half the vertical Angle.

Cor. 5. Rectangle of Sum of the three Sides, and Sum Legs — Baſe :
Rectangle Baſe + diff. Legs, and Baſe — diff. Legs : :
Radius Square :
Tangent Square of half the Angle at the Vertex.

This follows from Cor. 4. becauſe Coſine : Sine : : Rad : Tangent, by Prop. I. B. I.

SCHO-

SCHOLIUM.

Inſtead of the Rectangle of the Legs, you may take a Quarter of the Rectangle, of the Sum + the Difference of the Legs, into the Sum — diff. Legs; which is equal thereto.

Likewiſe inſtead of half the vertical Angle, you may take the Complement of half the Sum of the oppoſite Angles, which is alſo equal to it.

PROP. VIII.

In any Triangle AFB, *I ſay* 8.

$$AB - \frac{AF \times \text{Coſ. } A}{\text{Rad.}} : S.A :: AF : \text{Tan. oppoſite} \angle B.$$

For let fall the Perpendicular FR on the Baſe; then in the Triangle AFR, Rad. : AF :: S.A : FR $= \frac{AF \times S.A}{r} :: \text{Coſ. } A : AR = \frac{AF \times \text{Coſ. } A}{r}$. And $BR = AB - \frac{AF \times \text{Coſ. } A}{r}$, then BR : Rad : FR or $\frac{AF \times S.A}{r}$: Tan. B. Or BR : S.A :: AF : Tan. B.

Cor. $\sqrt{: AB^2 + AF^2 - \frac{2AB \times AF}{\text{Rad.}} \times \text{Coſ. } A :} = FB.$

For $FB^2 = AB^2 + AF^2 - 2AB \times AR$.

PROP. IX.

In the Triangle AFB, *draw* FR *perpendicular to the Baſe, and make* BR=RD; *then* 8.

As the Baſe, AB :
To the Difference of the Segments AD ::
So Sine of the vertical Angle F :
To Sine of the diff. vert. Angles, or S. diff. Angles at the Baſe B, A.

For AB : S.AFB :: FB or FD : S.A :: AD : S.AFD.

And $\angle B - \angle A = \angle FDB - \angle A = \angle AFD$.

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FIG.

PROP. X.

9. *As Coſ. diff. Angles at the Baſe — Coſ. Sum :*
Sine of the vertical Angle ::
So Perpendicular :
Half the Baſe.

For let A D E be the Triangle, D P the Perpendicular, C the Center of the circumſcribing Circle, draw E C G perpendicular, and D F parallel to A B, *&c.* the ∠ E C B (= 2 E B A) is the Sum of the Angles A, B, whoſe verſed Sine is E G. and E C D or 2 EBD is the Difference of the Angles A, B, whoſe verſed Sine is E F; therefore the Difference of theſe verſed Sines, or the Difference of the Coſines is F G or D P. Alſo G C B or A D B is the vertical Angle, whoſe Sine is G B. And therefore D P : G B :: diff. Coſines : S. vertical Angle.

Cor. As Sine of the diff. Angles at the Baſe :
Sine of the vertical Angle ::
Diſtance of the Perp. from the Middle of the Baſe :
Half the Baſe.

For F D is the Sine of the Difference of the Angles, A, B, and F D or G P : G B :: S. diff. : S. vert. Angle.

PROP. XI.

8. *As Radius :*
To Sum of the Cotan. half the Angles at the Baſe ::
So the Perpendicular :
To Sum of the three Sides.

Put $FR = p$, Radius $= r$, then by Prop. IV. Tan. $\frac{1}{2}A : r :: r : \text{Cot.}\ \frac{1}{2}A :: p : AF + AR$
and $r : \text{Cot.}\ \frac{1}{2}B :: p : BF + BR$
whence $AF + BF + AB = \frac{\text{Cot.}\ \frac{1}{2}A + \text{Cot.}\ \frac{1}{2}B}{r} p$.

PROP.

PROP. XII.

In a right angled Triangle ACB, FIG. 10.
As Radius :
Sine of double one acute Angle, A *or* B ::
Square of the Hypothenuſe :
Four times the Area.

Draw CD⊥AB, and biſſect the Hyp. AB in E, and draw EC, then AE=EC=EB, becauſe ACB may be inſcrib'd in a Semi-circle; then by ſimilar Triangles, $AB : BC :: AC : CD = \frac{AC+BC}{AB}$. Alſo $\angle CED = \angle B + BCE = 2B$, and $CEB = 2A$. Therefore by Prop. III. as Rad : CE or $\frac{1}{2}AB$ :: S.E : CD, or Rad : S.E :: $\frac{1}{2}AB : \frac{AC\times CB}{AB} :: \frac{1}{4}AB^2 : \frac{AC\times CB}{2}$ or the Area of the Triangle.

PROP. XIII.

As Radius :
To Sine of any Angle of a Triangle, A ::
So half the Rect. of the including Sides, $\frac{AF\times AB}{2}$: 3.
To the Area of the Triangle.

For the Area of the Triangle AFB is $\frac{1}{2}AB\times FH$; but Rad : AF :: S.A : $FH = \frac{AF\times S.A}{Rad.}$, therefore the Area $= \frac{AF\times AB\times S.A}{2\,Rad.}$

PROP. XIV.

As Coſ. diff. — Coſ. Sum of the Angles at the Baſe :
Sine of the vertical Angle :: 9.
So the Square of the Perpendicular :
To the Area of the Triangle ::

FIG. *And ſo the Area of the Triangle :*
9. *To the Square of half the Baſe.*

For by Prop. X. Coſ. $\overline{B-A}$— Coſ. $\overline{B+A}$: S.D :: DP : GB :: DP^2 : GB×DP, the Area :: GB×DP : GB^2.

PROP. XV.

3. *In any Triangle,*
As Radius :
Tan. half the vertical Angle ::
Rect. of half the Perimeter, and that half — Baſe :
To the Area of the Triangle.

For, let s=S.AFB ; c, t, τ the Coſine, Tangent, and Cotangent of half AFB, m=AF, n=FB, AB $=b$, $z=\frac{m+n+b}{2}$. Then by Prop. XIII. $\frac{mns}{2r}=$ Area, but by Sch. Prop. II. B. I. $s=\frac{2cc}{\tau}=\frac{2cct}{rr}$, therefore $\frac{mncct}{r^3}=$ Area. And by Cor. III. Pr. VII. $mncc=z\times\overline{z-b}$, therefore $\frac{z\times\overline{z-b}}{r}t=$ Area.

Cor. As Radius :
Cotan. half the vertical Angle ::
So Rect. Baſe +diff. Sides, and Baſe —diff. Sides:
Four Times the Area of the Triangle.

For if a=S. $\frac{1}{2}$ Angle F, then by Sch. Pr. II. B. I. $s=\frac{2aa}{t}=\frac{2aa\tau}{rr}$, and the Area $=\frac{mns}{2r}=\frac{mna^2\tau}{r^3}$. But by Cor. 3. Prop. VII. $mna^2=\overline{z-m}\times\overline{z-n}\times rr=\frac{b+n-m}{2}\times\frac{b+m-n}{2}r^2$, therefore the Area $=\frac{b+n-m}{2}\times\frac{b+m-n}{2r}\tau$.

PROP. XVI.

If three Lines be drawn from the three Angles of a Triangle to any Point O, *the product of the Sines of all the alternate Angles will be equal.* 9.

Let the Sines of the Angles be as in the Figure, then in the Triangle

$$\left.\begin{array}{llll} \text{POI,} & a : f & :: \text{IO} : \text{PO} \\ \text{IOH,} & e : d & :: \text{HO} : \text{IO} \\ \text{HOP,} & c : b & :: \text{PO} : \text{HO} \end{array}\right\} \text{theſe multiply'd}$$

$$aec : fdb :: \text{IO}\times\text{HO}\times\text{PO} : \text{PO}\times\text{IO}\times\text{HO}.$$

but the two laſt Terms are equal, and therefore the two firſt, $aec = fdb$.

Cor. 1. If two Angles, P, H, are biſſected, by the Lines P O, H O; then the third Angle I will alſo be biſſected. For ſince $ace = bdf$, and $a = b$, and $c = d$, or $ac = bd$, therefore $e = f$; and the contrary.

Cor. 2. Hence alſo three Perpendiculars erected on the middle of the three Sides of a Triangle, interſect in one Point. And the contrary.

For if two Perpendiculars be erected on two Sides P H, I H, then will $b = c$, and $d = e$, whence $a = f$, therefore the Perpendicular ſtands on the Middle of P I.

SCHOLIUM.

If the Point O, be taken out of the Triangle, this Prop. will hold equally true.

PROP. XVII.

If four right Lines be drawn from the four Angles of a Trapezium, to any Point in it as O; *the Products of the Sines of the alternate Angles, will be equal to each other.* 12.

FIG. In the Triangle

12. SPO, $a : b :: SO : PO$.

POH, $c : b :: PO : HO$

HOI, $e : d :: HO : IO$

IOS, $g : f :: IO : SO$

their Product

$$aceg : bbdf :: SO.PO.HO.IO : PO.HO.IO.SO.$$

therefore

$$aceg = bbdf.$$

Cor. 1. After the ſame Manner, if Lines be drawn from all the Angles of any Polligon whatever to a Point; the Products of the Sines of the alternate Angles will be equal.

13. *Cor.* 2. In any Trapezium, let a, b, c, d *&c.* be the Sines of the Angles, as in the Figure; then

As $bbd+ace$:

$bbd-ace$ ::

Tan. half the Sum of the Angles, $\frac{ISH+SIP}{2}$:

Tan. half the Difference $\frac{ISH-SIP}{2}$.

For ſince $aceg=bbdf$, therefore $ace : bbd :: f : g :: SO : IO$, and by Compoſition and Diviſion, $bbd+ace : bbd-ace :: IO+SO : IO-SO ::$ (by Prop. VI.) Tan. $\frac{ISO+SIO}{2}$: Tan. $\frac{ISO-SIO}{2}$.

SECT. II.

The Solution of all the Caſes of plain Triangles.

From the foregoing Propoſitions, all the Caſes of plain Triangles may be reſolved. Every Triangle has 6 Parts, 3 Sides and 3 Angles; and any three Things being given (except the 3 Angles) the other 3 may be found; but if only 3 Angles are given, there may be found an infinite Number of Triangles, that will have theſe three Angles. I ſhall here give the Solution of right Angled-triangles 3 different ways, *viz.* arithmetically, logarithmically, and algebraically: and that of oblique Triangles, logarithmically and algebraically. For it is needleſs to ſpend time in ſolving theſe arithmitically; ſince any oblique Angled-triangle is divided into two right ones by a Perpendicular, and then they are reſolved by the Caſes of right Angled-triangles; this Method is true to 3 or 4 places of Figures, and is ſufficient in common Caſes. I have alſo omitted their Solution by Projection and inſtrumentally, as being only Approximations to the Truth; and they are beſides ſo eaſiy in themſelves, to any body that can but handle a Pair of Compaſſes, as to need no particular Explication. I ſhall only obſerve in general, with Regard to *Gunter*'s Scale, *&c.* That any ſimple Proportion may be wrought on it by this *general Rule.*

Extend the Compaſſes from the firſt Term to one of the Means, on its proper Line; that extent, ſet the ſame way, upon its proper Line, will reach from the other Mean to the fourth Term required; where Radius is the Sine of 90°, or Tan. 45°. If, in extending upon the Tangents, the Compaſſes reach beyond the End; ſet it ſo far back as it reaches over.

Now according to the different Variety of things given and ſought, the Solution of Triangles is divided into ſeveral Caſes as follows; in which I have only put the eaſieſt Solutions. They that require more may conſult the Propoſitions, from whence they are derived. I have added no numerical Examples, becauſe any body can add and ſubtract, according to *this Rule*.

Add the Logarithms of the ſecond and the third Terms together, from which ſubtract the Logarithm of the firſt Term, and the remainder is the Logarithm of the fourth Term ſought, where Radius is 10.

But in algebraic Solutions you muſt uſe natural Sines, Tangents *&c.* and they muſt be actually multiply'd, and divided, and here Radius is 1.

Note, In right Angled-triangles if one acute Angle is given, the other is found by ſubtracting this from 90.

And in oblique Triangles, if 2 Angles be given, the third is found by ſubtracting their Sum from 180. or ſubtracting one from 180 gives the Sum of the other two.

In ſolving any of the Caſes, it will be proper for diſtinction's ſake to mark what is given with a Daſh (/), and what is ſought with a Cypher (o).

◆

Right angled plain Triangles.

CASE I.

An Angle C and the Hypothenuſe BC, being given; to find a Side, BA. FIG. 14.

I. Arithmetically without Tables.

Let D= Degrees in the leſſer Angle; then

$$\frac{57\cdot3}{D}+\frac{3D}{1000} : 1 :: \text{Hyp. BC : leſſer Side CA.}$$

$$\text{or} \quad 1 : 1-\frac{1\frac{1}{2}DD}{10000} :: \text{Hyp. BC : greater Side BA.}$$

II. Logarithmically, by the Table of artificial Sines &c.

Rad : Hyp. BC :: S. an Angle C : op. Side BA.

III. Algebraically, by the Table of nat. Sines &c.

Let Hyp. BC=b, nat. Sine of C=s, Rad=1. then

$1 : b :: s : sb=$BA.

CASE II.

Given an Angle B or C, and a Leg; to find the Hypothenuſe, BC. 14.

I. Arithmetically.

Let D= Degrees in the leſſer Angle, then

$$1 : \frac{57\cdot3}{D}+\frac{3D}{1000} :: \text{leſſer Side AC : Hyp. BC.}$$

$$\text{or} \quad 1-\frac{1\frac{1}{2}DD}{10000} : 1 :: \text{greater Leg BA : Hyp. BC.}$$

II. Logarithmically.

S. Angle B ; oppoſite Side CA :: Rad : Hyp. BC.

III. Alge-

FIG.

III. Algebraically.

14. Let s=S.B, p=AC, then $\frac{p}{s}$= Hyp. BC.

CASE III.

14. Given an Angle, and a Leg CA; to find the other Leg, BA.

I. Arithmetically.

Find the Hyp. BC, by Caſe II. then $\sqrt{BC^2 - CA^2}$ = BA.

II. Logarithmically.

Rad : one Leg CA :: Tan. Angle C : op. Side BA.

III. Algebraically.

Let S.C=s. S.B=c, Tan. C=t. CA=p, then pt=BA, or $\frac{ps}{c}$=BA.

CASE IV.

14. Given the Hyp. BC, and a Leg CA; to find an Angle B, or C.

I. Arithmetically.

Find the other Leg, BA, by Caſe V. then As Hyp. BC+½ the longer Leg BA : ſhorter Leg CA :: 86 : op. Angle B, and B taken from 90 gives C.

II. Logarithmically.

As Hyp. BC : Rad :: given Leg CA : S. op. Angle B, and B taken from 90 gives C.

III. Algebraically.

Let Hyp. BC=h, Leg CA=p, then $\frac{p}{h}$=S.B, and 90−B=C.

CASE V.

Given the Hypothenuſe B C, and a Leg C A, to find the other Leg B A. FIG. 14.

I. Arithmetically.

$$\sqrt{BC^2 - CA^2} = BA.$$

II. Logarithmically.

Take the Sum and Difference of the Hyp. B C, and given Leg A C, add their Logarithms together; half the Sum is the Logarithm of the Side required, A B.

III. Algebraically.

Let $BC = b$, $CA = p$, then $\sqrt{bb - pp} = BA$.

Or let $BC + AC = z$, $\overline{BC - AC} = d$, then $\sqrt{dz} = BA$.

CASE VI.

The Legs A B, A C, being given to find an Angle, B. 14.

I. Arithmetically.

Find the Hyp. B C by Caſe VII. then as Hyp. B C + half the longer Leg, B A : leſſer Leg C A : : 86 : Angle op. B.

II. Logarithmically.

As one Leg A B : Rad : : other Leg A C : Tan. B, the op. Angle.

III. Algebraically.

Let $AB = b$, $AC = p$, then $\frac{p}{b} =$ Tan. B or $\frac{p}{\sqrt{bb + pp}}$ = S.B.

FIG. 14.

CASE VII.

The Legs A B, A C being given; to find the Hypothenuſe B C.

I. Arithmetically.

$$\sqrt{AB^2+AC^2}=BC.$$

II. Logarithmically.

Find the Angle B by Caſe VI. and then the Hyp. B C by Caſe II.

III. Algebraically.

Let $AB=b$, $AC=p$, then $\sqrt{bb+pp}=BC$.

Oblique plain Triangles.

CASE I.

Given two Angles A, B, and an oppoſite Side, FB; 15.
to find the other oppoſite Side A F. 16.
17.

I. Logarithmically.

S.one Angle A : op. Side F B :: S.the other Angle B : to its oppoſite Side A F.

II. Algebraically.

Let a=S.A, s=S.B, F B=d then $\frac{sd}{a}$ = A F.

CASE II.

Two Sides A F, F B, and an oppoſite Angle B 15.
being given; to find the other oppoſite Angle A. 16.
17.

I. Logarithmically.

One Side A F : S.its op. Angle B :: other Side F B : S. of its op. Angle A.

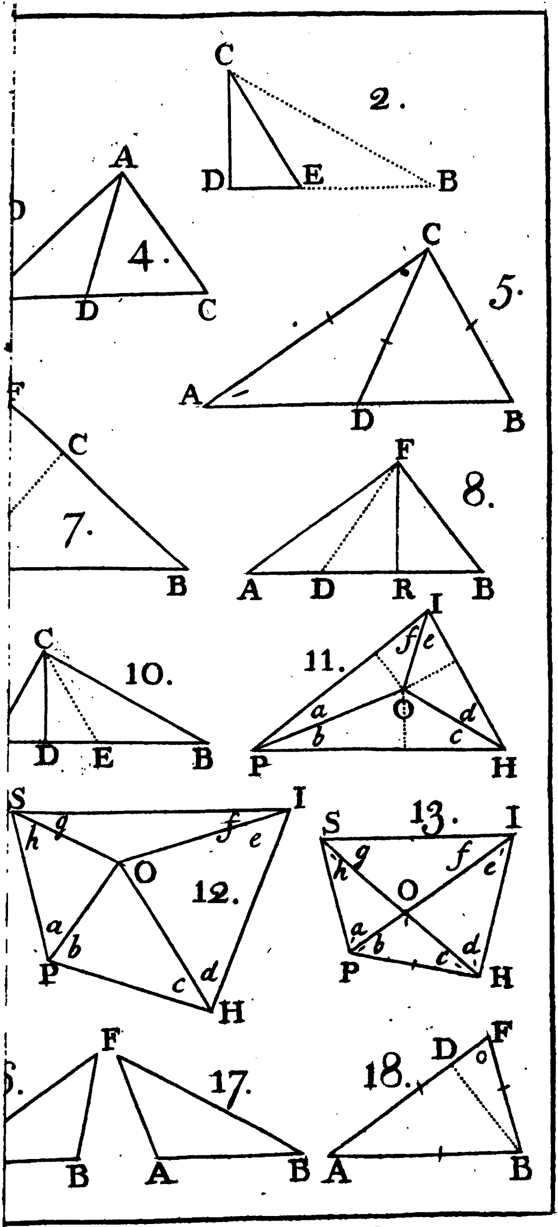
If the given Angle B is obtuſe, or if its oppoſite Side A F be greater than the other Side B F, then Angle A ſought is acute; otherwiſe 'tis doubtful: And the Sum of A and B taken from 180 gives F.

II. Algebraically.

Let s=S.B, A F=c, F B=d, then $\frac{sd}{c}$ = S.A.

SCHOLIUM.

In the Solutions of Plane Triangles there are three Caſes, where the Thing ſought requires two Operations, 1. When two Legs of a right angled Triangle are given to find the Hypothenuſe, 2. When two Sides and the included Angle, in an oblique Triangle, are given, to find the third Side, 3. When two Sides and an Angle oppoſite are given, in an oblique Triangle, to find the third Side.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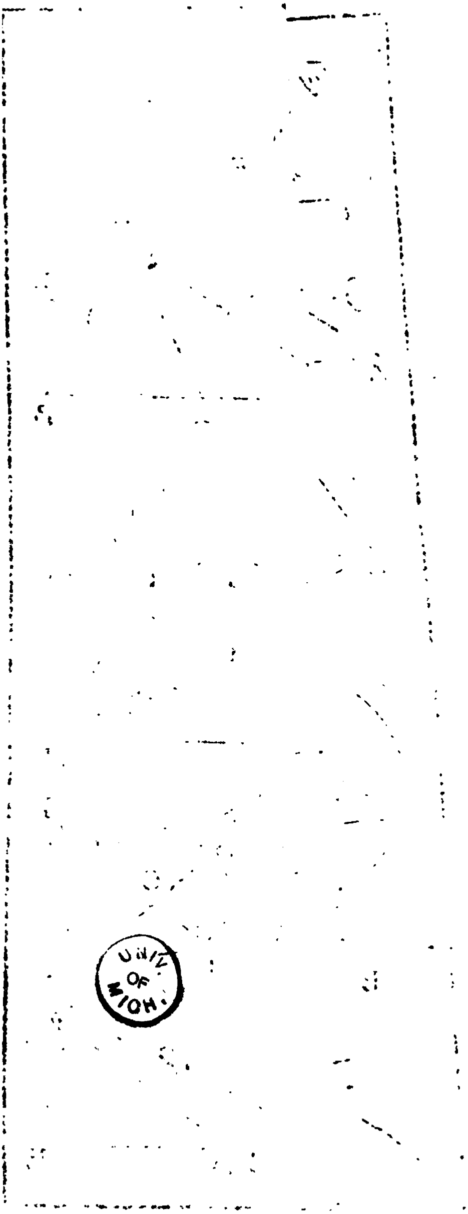

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VI

THE
ELEMENTS
OF
TRIGONOMETRY.

BOOK III.

The Doctrine of the Sphere and ſpherical Trigonometry.

DEFINITIONS.

1. *SPherical Trigonometry*, teaches the Properties of ſpherical Triangles, and how to calculate their Sides and Angles.

2. *A Circle of the Sphere* is that which is made by a Plane cutting the Surface of the Sphere. If the Plane paſs thro' the Center, 'tis a great Circle: If not, it is a leſſer Circle.

3. *The Pole of a Circle* is a Point on the Surface of the Sphere equidiſtant from every Point of the Circle. Every Circle has two Poles, diametrically oppoſite to each other.

4. *A ſpherical* Angle is the Inclination of two Circles of the Sphere to one another, interſecting in a Point call'd the angular Point.

5. *A right Angle* is 90 Degrees; an acute Angle is leſs, and an obtuſe Angle greater than a right one.

6. *A ſpherical Triangle* is made upon the Surface of the Sphere, by the Interſection of three great Circles.

If one Angle be right, 'tis called a *right angled* ſpherical Triangle. If one Side be a Quadrant 'tis called a *quadrantal* Triangle. If no Angle or Side be 90°, 'tis an oblique Triangle.

7. In a right angled Triangle, the Side oppoſite to the right Angle is called the *Hypothenuſe*; the other two are called *Legs* or Sides.

8. In a right angled or quadrantal Triangle, the five circular Parts (ſetting aſide the right Angle or quadrantal Side) are, the two Parts adjoining on each Side the right Angle or quadrantal Side, and the Complements of the three Parts which are furtheſt diſtant from it. The *middle Part* is that which is equidiſtant from other two; which two if they adjoin to the middle Part are called *extreams conjunct*; and if they be remote from it, are called *extreams disjunct*.

9. Sides and Angles are ſaid to be of the *ſame Affection*, when both are greater, or both leſſer, or both equal to 90°. And they are of *different Affection*, if one is greater and another leſs than 90°.

10. *Like or ſimilar Arches* of different Circles are thoſe that contain the ſame Number of Degrees.

* * *

SECT. I.

The Properties of ſpherical Angles and Arches.

PROP. I.

The Section of a Sphere and a Plane is a Circle.

FIG. 19.

Let BIC be the Section, and let the Plane of the great Circle ABCD be drawn thro' the Center E of the Sphere, perpendicular to the Plane BIC, and let BHC be their common Interſection. Draw EH perpendicular to BC, then BH=HC. Take any Point I in it, and draw HI, IC; and alſo BE, CE. Since EH is perpendicular to BC, and in the perpendicular Plane ABCD, therefore 'tis perpendicular to the Plane BIC, and therefore to HI. But BE, CE, IE are equal Radii of the Sphere; therefore in the right angled Triangles BEH, IEH; $BH^2+HE^2=BE^2=IE^2=HI^2+HE^2$, therefore $BH^2=HI^2$, and BH=HI, therefore BIC is a Circle whoſe Center is H.

Cor. 1. If a great Circle ABCD be perpendicular to any other Circle BIC it paſſes thro' its Poles.

For ſince it is prov'd that H is the Center of the Circle BIC, and EH perpendicular to it, produce EH to P which will be in the Plane of the Circle ABD, becauſe EH is in it. Draw the great Circle PI, then ſince the Sines BH, HI, HC are equal, therefore the Arches belonging to them BP, IP, CP are equal, therefore P is the Pole. And if PE was produced to the other Pole, that Pole will alſo be in the Plane of the Circle ABC.

Cor. 2. If a great Circle ABC paſſes thro' the Pole P of another Circle, it cuts it at right Angles and into two equal Parts.

FIG. 19. For draw P E to cut the Circle in H, then is H the Center, and P H perpendicular to its Plane, and conſequently EH and the Plane ABCD are perpendicular to it. And ſince the Center H is in B C, the common Interſection of the Planes; therefore B I C is a Semicircle.

PROP. II.

20. *All great Circles of the Sphere cut one another into two equal Parts.*

For the common Section of their Planes is the Diameter of the Sphere, and conſequently their Segments are Semicircles.

Cor. 1. Any Side of a ſpherical Triangle is leſs than a Semicircle.

For ſince A D C or A B C is a Semicircle, therefore in the Triangle A D B, A D or A B is leſs than a Semicircle.

Cor. 2. If two Sides of a ſpherical Triangle A B D be produced, till they interſect in C, each will become a Semicircle.

PROP. III.

21. *If one or more Circles interſect another Circle in one and the ſame Point* C; *the Sum of the Angles* A C E, E C D *is equal to two right Angles.*

Let the Arch E C interſect A D in C, and draw C B perpendicular to A D; then $ACB+BCD=$ two right Angles, but $ACE+ECB=ACB$, and $ECD-ECB=BCD$, therefore $ACE+ECD=ACB+BCD=$ two right Angles.

Cor. 1. All the Angles made about one Point are $=$ four right Angles.

Cor. 2. When one Circle croſſes another, the vertical Angles a, c, are equal.

For $a+b=$ two right Angles, and $b+c=$ two right Angles, therefore $a+b=b+c$ and $a=c$.

PROP. IV.

A ſpherical Angle made by two great Circles is meaſur'd by the Arch of a great Circle intercepted between the Sides, and deſcribed at a Quadrant's diſtance from the Angular Point. 22.

Let $AB=AD=90^\circ$, and draw the Arch BD, and from the Center E draw BE, DE; alſo draw the Tangents AT, AS, to the Arches AB, AD; then the $\angle TAS=$ ſpherical Angle BAD, and AEB, AED will be two right Angles, but TAE, SAE are right Angles; therefore TA, SA are $\parallel$ to BE, DE, and $\angle TAS=BED$. But the Arch BD is the Meaſure of the Angle BED or TAS, and therefore of the ſpherical Angle BAD.

Cor. 1. The Angles BAD, BCD, made at oppoſite Points of the Semicircle are equal. For the $\angle$ BED or Arch BD is the Meaſure of both.

Cor. 2. The Diſtance of the Poles of two Circles is equal to the Angle of their Inclination.

For ſince AE is $\perp$ to the Plane BDE, this Plane is $\perp$ to both the Planes ABC, ADC; therefore the Poles are in the Circle BD, as ſuppoſe at P, Q. Then $BP=90^\circ=DQ$, and ſubtracting DP from both, then $BD=PQ$.

Cor. 3. Two great Circles ABC, ADC paſſing thro' the Poles of another great Circle BD, will cut all the Parallels to BD, as *bd*, into ſimilar Arches.

For AC is $\perp$ to *bed*, and *e* the Center of *bd*, theretherefore $\angle BED=bed$, or the Arch *bd* ſimilar to BD.

Cor. 4. Hence an Angle made by two great Circles of the Sphere, is equal to the Angle of Inclination of the Planes of theſe great Circles. For $\angle$ BED or $bed=\angle$ TAS or BAD.

FIG. 22. *Cor.* 5. Hence alſo if *b d* be a Parallel to the great Circle B D P; it is as Radius : Coſine of the Parallel's diſtance from its great Circle : : ſo any Arch of the great Circle : to a ſimilar Arch of the Parallel.

For B E : B D : : *be* : *bd*.

PROP. V.

23. *If any two Circles* A E B, A F B, *interſect one another on the Sphere, they make the oppoſite Angles* A *and* B *equal, and the oppoſite Parts* A F E, B F E *ſimilar.*

Let A B be the common Section of their Planes; and in the Planes of their reſpective Circles, draw the Tangents A C, B C to the Circle A F B; and A D, B D to the Circle A E B, and join C D. Then in the Triangles A C D, B C D, $AC = BC$, being Tangents to the ſame Circle and for the ſame Reaſon $AD = BD$, and C D is common; therefore the Triangles are equal and ſimilar, and the $\angle CAD = CBD$, and therefore the Spherical Angles A and B equal to them, are alſo equal to one another.

Likewiſe the oppoſite Parts of the Figure A F B E A, are ſimilar, being bounded by Parts of the ſame Circles, having the ſame Poſition.

Cor. Hence any two like Arches O I, S V, drawn after a like manner, will be equal. That is if $AO = BS$, and $AI = BV$, or $\angle AOI = \angle BSV$, then $OI = SV$. For then the Triangles A O I, and B S V are ſimilar and equal.

SCHOLIUM.

Tho' the oppoſite Parts of the Figure A F E and B F E are ſaid to be ſimilar; yet if the Angle A be apply'd to B, the Triangles A F E, B F E will not coincide. Or if the Baſe F E of one be laid upon the Baſe E F of the other, the triangular Figures A F E, B F E will lye contrary ways, one to the right, the other to the left hand.

PROP.

FIG.

PROP. VI.

If two great Circles H O, H R *have the ſame Inclination to a third* EQ; *the Arches of all the Parallels* (AP) *to the third, intercepted by the other two, will be ſimilar Arches.* 24.

Let the great Circle E C, cut the Circles H O, H R at equal Angles in E, and Q. Thro' the Poles of E C draw the great Circles P B, A C. Then in the Triangles P E B, A Q C, $\angle E = \angle Q$, and the Angles at B and C are right, and PB = A C: therefore if the Angle E be laid upon the Angle Q, then will P fall ſomewhere in the Arch P A (becauſe P B = A C); and ſomewhere in the Arch QH (becauſe $\angle E = \angle Q$), therefore P falls upon A, their Point of Interſection; and conſequently B upon C (otherwiſe A would be the the Pole of C Q); therefore B E = C Q, and P E = A Q, therefore E Q = B C; but B C is ſimilar to A P (by Cor. 3. Prop. IV), therefore E Q and A P are ſimilar.

Cor. The ſame Things ſuppoſed, the Arches of the two inclined great Circles intercepted between any two Parallels to the third, are equal, P E = A Q.

PROP. VII.

If a Plane E P Q *be ſuppoſed to be drawn perpendicular to the Planes of two equal Circles* HO, EQ; *and any other Plane* P D K N *be drawn thro' the two Poles,* P, N, *which are equidiſtant from their reſpective Circles, and to revolve about* P N: *Then theſe Planes will intercept equal Arches of theſe Circles,* D O, K Q. 25.

For let N be the Pole of H O, and P of EQ, and N O = P Q, then by Cor. Prop. V. $\angle$N O H or NOD = P Q E or P Q K; and $\angle$ P O D = N Q K, and N Q = P O, and by Prop. V. $\angle N = P$. Therefore the Triangles P O D and N Q K are ſimilar and equal, and

FIG. and therefore Arch DO=KQ. And likewiſe Arch HD=EK.

PROP. VIII.

26. *Of ſeveral Arches of great Circles drawn from the ſame Point* A *to any other Circle* CED; *the greateſt* AD *is that which paſſes thro' the Pole* P, *and the nearer to this as* AB *is greater than that which is further off as* AE. *And the leaſt is* AC *the remainder of that which paſſes thro' the Pole.*

Let AS be perpendicular to the Plane of the Circle CED, O its Center, CSOD its Diameter. Draw the right Lines SB, SE, AD, AB, AE, AC. Then you have the right angled plane Triangles ASD, ASB, ASE, ASC, wherein the Perpendicular AS is common. But (Eucl. III. 7.) the Baſe SD paſſing thro' the Center is the greateſt, and SB greater than SE, and SE greater than SC which is the leaſt.

Therefore the Hypothenuſe AD is the greateſt, AC the leaſt, and AB greater than AE. But theſe are the Cords of Arches, and to greater Cords belong greater Arches, conſequently the Arch AD is the greateſt, Arch AB (nearer to AD) is greater than Arch AE (further from it); and Arch AC the leaſt of all.

27. *Cor.* 1. If two Sides in two ſpherical Triangles APE, APB, be reſpectively equal; that which hath the greater included Angle hath the greater Baſe, and the contrary.

For about the Pole P, at the Diſtance PE or PB (the greater *ſide* in the two Triangles), deſcribe the Circle EBD, and from A thro' P draw the great Circle APD. Then by this Prop. if BD be leſs than ED, or ∠APB greater than APE; then is AB greater than AE.

And if AB be greater than AE, then AB is nearer to D than AE, ∠APB greater than APE.

Cor. 2. The Perpendicular let fall on the Baſe of a ſpherical Triangle, is either greater than either Side, or leſs than either Side.

For it paſſes thro' the Pole of the Baſe, and is therefore either the longeſt or ſhorteſt Line drawn from the Vertex of the Triangle.

Cor. 3. And therefore when the Perpendicular falls without the Triangle, either the greater or leſſer Perpendicular may be eſteem'd the Perpendicular upon the Baſe.

28. *Cor.* 4. Hence if the Perpendicular fall without, as in the Triangle B A E, the greater Perpendicular A D lyes next the greateſt Side A B; and the leaſt Perpendicular A C next the leaſt Side A E.

But when the Perpendicular falls within; if it is leſs than a Quadrant as A C (in the Triangle A E F), then it lyes nearer the leſſer Side A E, and the Segment C E is leſs than C F, and ∠ C A E leſs than C A F. But if it is greater than a Quadrant, as A D (in the Triangle A B G), then it lyes neareſt the greatter Side A G, and then it makes the leſſer Segment of the Baſe (D G), and leſſer vertical Angle (D A G), next the greater Side A G. And the greater ones next the leſſer Side A B.

Cor. 5. The Perpendicular falling within, or the neareſt Perpendicular falling without, is of the ſame Affection, as half the Sum of the Sides.

For in the Triangle E A F, A C is leſs than a Quadrant, and ſince $AF < AG$, therefore $AE + AF < EG$ or 180, and $\frac{AE+AF}{2} < 90$. But in the Triangle G A B, A D is $>$ a Quadrant; and ſince $AG > AF$, therefore $BA + AG > BF$ or 180, and $\frac{BA+AG}{2} > 90$.

Again in the Triangle B A E, where the Perpendicular A C is leſs than a Quadrant, and $CE < CF$; then by this Prop. $AE < AF$, and $AE + AB < BF$ or 180°, and $\frac{AE+AB}{2} < 90$. But in the Triangle G A F, where the Perpendicular A D (greater

than

FIG. 28. than a Quadrant) is neareſt, or $DB > DG$, or $CF > CE$, then $AF > AE$, and $AF+AG > GE$ or 180. and $\frac{AF+AG}{2} > 90$.

Cor. 6. Hence when the Perpendicular falls within, or you take the neareſt Perpendicular falling without; then if half the Sum of the Sides be leſs than a Quadrant, the leſſer Segment, and leſſer vertical Angle adjoins to the leſſer Side; but if half the Sum of the Sides be greater than a Quadrant; the leſſer Segment and leſſer vertical Angle adjoins to the greater Side.

SECT. II.

The Affections of ſpherical Triangles,

PROP. IX.

FIG. 29.

Every Triangle ABC *hath (by producing its Sides) another Triangle* abc*, on the oppoſite Side of the Globe, ſimilar and equal to it.*

For the Point a is oppoſite to A, or a Semicircle diſtant from it; and b is oppoſite to B, and c to C. Therefore $ab = AB$, $bc = BC$, and, $ac = AC$. Likewiſe by Cor. 1. Prop. IV. $\angle a = \angle A$, $b = B$, and $c = C$; therefore the Triangle abc is equal and ſimilar to ABC.

Cor. I. Hence any Triangle ABC, by producing its Sides, hath oppoſite to every Angle thereof, another Triangle of the ſame Baſe and oppoſite Angle with the former; and the other Parts the Supplements thereof.

Thus in the Triangle Bca, $\angle B = \angle ABC$, $ac = AC$, aB = Sup. AB, cB = Sup. CB, $\angle ca$B = Sup. CaB or A, and $\angle ac$B = Sup. C.

Cor. 2. Any Triangle ABC, has adjoining to every Side thereof another Triangle having the ſame Side and oppoſite Angle, and the other Parts the Supplements thereof.

Thus in the Triangle ABc, AB and $\angle c$ is the ſame with AB and $\angle C$; Bc, Ac are the Supplements of BC, AC; $\angle c$BA = Sup. CBA, and $\angle c$AB = Sup. CAB, Likewiſe there is the Triangle aBC adjoining to CB; and the Triangle AbC adjoining to AC.

SCHO-

FIG. 29.

SCHOLIUM.

Tho' the Triangles ABC, *abc* are ſaid in this Prop. to be ſimilar; yet they will not coincide when apply'd to one another. For laying any Angle *a* upon its equal A, ſo that the Concavities of the Triangles lye the ſame way; the Side *ac* will fall on AB, and *ab* on AC.

PROP. X.

30. *If from the three Angles of any ſpherical Triangle* ABC, *as Poles, you deſcribe three great Circles, they will form another ſpherical Triangle* DEF *by their Interſections; each Side and Angle whereof will be the Supplements of the Angle and Side oppoſite, in the given Triangle.*

Produce the Sides as in the Fig. then ſince Bk or BF is a Quadrant, and Ag or AF is a Quadrant, therefore F is the Pole of AB; and likewiſe E the Pole of AC, and D the Pole of CB. then $DQ=$ a Quadrant $=ER$, and $DE=QR=$ Sup. $\angle C$.

Alſo $Eg=90^\circ=Fp=Tp$. and $TE=pg$, therefore $FE=$ Sup. ET or pg that is of the $\angle CAB$. Likewiſe $DT=kr$ or $\angle B$, and $DF=$ Sup. $\angle B$.

Alſo $rB=90^\circ=Ap$, and $rB+Ap$ or $rp+AB=180^\circ$, and rp or the $\angle F=180-AB=$ Sup. AB.

And thus $Ag=CR$, and $AC=gR$, and Sup. $gR=$ Sup. AC, that is $\angle E=$ Sup. AC.

Laſtly $CQ=90^\circ=Bk$, and $CQ+Bk$ or $kQ+CB=180$, and $kQ=180-CB$, that is $\angle D=$ Sup. CB.

Cor. 1. Some three Poles of the Sides of any Triangle, form another Triangle, wherein the Angles and Sides are reſpectively equal to the Supplements of the Sides and Angles of the other; and theſe two Triangles are mutually ſupplemental to each other.

For D, E, F are the Poles of the Sides of the Triangle

angle A B C, as well as A, B, C are the Poles of the Sides of D E F. FIG. 30.

Cor. 2. The neareſt Poles D, E, T of the Sides of any ſpherical Triangle A B C, form another Triangle, wherein the Angles and Sides are reſpectively equal to the Sides and Angles of the given one: Excepting only the Supplements of one Side and its oppoſite Angle in the former, will be the correſpondent Angle and Side in the latter.

For in the Triangles D F E, D T E, D E is common, and ∠ F = ∠ T; and all the Reſt are the Supplements. There are other five, or in all ſix ſuch Triangles, and all adjoining to the Angles and Sides of the Triangle D E F, by Cor. 1, 2, Prop. IX. but there is only one ſupplemental Triangle D E F to the Triangle A B C.

SCHOLIUM.

By this Prop. quadrantal Triangles may be reduced to right angled ones.

PROP. XI.

In two Triangles, if three Sides in one be reſpectively equal to three Sides in the other; or the three Angles in the one, to the three Angles in the other; Theſe Triangles will be equal in all reſpects.

Caſe 1. If three Sides be equal; and if the Baſe of one Triangle be laid on the Baſe of the other, the two other Sides muſt coincide; becauſe there can be but one Point from whence two given Arches can be drawn to the two Angles at the Baſe.

Caſe 2. If three Angles are equal, then by Cor. 1. Prop. IX. the Sides of two Triangles form'd by the Poles of both, will be the Supplements of the Angles of the former, and therefore reſpectively equal. Therefore by Caſe 1 the Angles of theſe ſupplemental Triangles are equal, and theſe Angles are the Supplements of the Sides of the former Triangles by Cor. 1.

FIG. Prop. X. therefore the Sides of the former Triangles are reſpectively equal.

SCHOLIUM.

This Demonſtration ſuppoſes the equal Sides to lye the ſame way; but if they lye contrary ways, they cannot concide. Yet if you will ſuppoſe the Convexity of one of them to lie the contrary way, then they will coincide when laid upon one another. The ſame may be apply'd to the following Prop.

PROP. XII.

If in two Triangles, there be two Sides and the included Angle; or two Angles and the included Side, reſpectively equal; the two Triangles will be equal in all reſpects. For,

1. If one ſide in one Triangle be laid upon its Equal in the other Triangle; then by reaſon of the included Angle being equal in both, the other two Sides will coincide, and ſo the whole Triangle.

2. If the equal Side in one be laid upon that in the other, and the equal Angles upon one another, the whole Triangles will coincide, and will therefore be equal.

PROP. XIII.

The Sum of any two Sides of a ſpherical Triangle is greater than the third Side.

31. Let A B be the greateſt Side; about the Pole B, thro C deſcribe the Arch C D, which will be ⊥ to B D, alſo about the Pole A thro' D deſcribe the Arch DE, which will alſo be ⊥ to A D. And ſince the two Circles C D, E D, can but touch in one point D, therefore at E, C there is a ſpace EC between them. Whence, ſince $AE+CB=AD+DB$ or A B, therefore $AE+EC+CB$ or $AC+CB$ is greatter than AB.

Cor. 1. The Sum of the three Sides of any ſpherical Triangle D F E is leſs than a Circle. FIG. 30.

For D E is leſs than D T+T E, and therefore D E+E F+F D is leſs than F D T+F E T or two Semicircles.

Cor. 2. A great Circle is the neareſt Diſtance of any two Points upon a Sphere. For it is a Line of the leaſt Curvature that can be drawn, or deflects leaſt from a right Line.

PROP. XIV.

In any Triangle equal Sides are oppoſite to equal Angles. 32.

1. Let A C=B C, let C D biſſect the Angle A C B, then in the Triangles A C D, C D B, there are two Sides and the included Angle at C reſpectively equal; therefore by Prop XII, the $\angle$ A=$\angle$ B.

2. If $\angle$ A=$\angle$ B, make A E=B E, then by Caſe 1. the $\angle$ E A B=$\angle$ B=$\angle$ C A B by Suppoſition; therefore E falls upon C, and A C is=B C.

Cor. 1. An equilateral Triangle is alſo equi-angular, and the contrary.

Cor. 2. In an Iſoceles Triangle, a Perpendicular let fall on the Baſe, biſſects it and the vertical Angle: *et contrà.*

Suppoſe the Convexity of the Triangle C D B to lye the contrary way; then becauſe the Angles at D are equal; if C D B be laid upon C D A, D B will fall upon D A; and becauſe C B=C A, and $\angle$ A=$\angle$ B (by this Prop.), therefore the Point B will fall upon A; and the Triangles will coincide, then will A D=D B, and $\angle$'s at C will be equal.

Again, If A D=D B, or $\angle$ A C D=$\angle$ B C D, then by Prop. XII. the Angles at D are equal, or C D$\perp$A B.

PROP. XV.

31. *In any ſpherical Triangle, the greateſt Side is oppoſed to the greateſt Angle; and the leaſt to the leaſt.*

1. Let $\angle C > B$, and make $\angle BCD = B$. then will $CD = BD$ by Prop. XIV. but $CD + AD$ or $AB > AC$ by Prop. XIII. For the ſame Reaſon if $B > A$, then $AC > CB$.

2. Let $AB > AC$, then Angle $C > B$; for if C be equal or leſs than B, then A B is equal or leſs than C A by Caſe 1, which is contrary to the Suppoſition.

PROP. XVI.

33. *If one Side* D A *of a Triangle* D A B *be produced, and if the Sum of the other ſides* $DB + BA$ *be greater, equal, or leſs than a Semicircle; the internal Angle at the Baſe* D, *is accordingly greater, equal or leſs than the outward oppoſite Angle* B A C.

For if $DB + BA$ be $>$ $=$or $<$ than D C, then B A is $>$ $=$or $<$ than B C; and by Prop. XV. the Angle C or D is $>$ $=$ or $<$ than BAC, reſpectively.

Cor. 1. In an Iſoceles Triangle, as one of the equal Sides is greater, equal, or leſs than a Quadrant; the Angle at the Baſe is accordingly greater, equal, or leſs than a right Angle.

Cor. 2. In an Iſoceles Triangle, either Side is of the ſame Affection as its oppoſite or adjacent Angle.

PROP. XVII.

33. *In any ſpherical Triangle, if the ſum of any two Sides be greater, equal, or leſs than a Semicircle; the Sum of the oppoſite Angles, is accordingly greater, equal or leſs than two right Angles.*

For by Prop. XVI, if DB+BA be greater, equal or leſs than DC, then ∠D is greater, equal, or leſs than B A C, and D+D A B greater equal, or leſs than D A B+B A C or two right Angles. FIG. 33.

Cor. 1. In any Triangle half the Sum of any two Sides is of the ſame Affection as half the Sum of their oppoſite Angles.

Cor. 2, Hence it follows, that if in a Triangle A C B, there be given two Sides A C, C B, and an oppoſite Angle A; and if C D be made=C B, then you will have two Triangles A C B, A C D containing the Things given, and therefore the other oppoſite Angle may be either B or A D C; and conſequently in both of them the Sum of the Sides A C, B C; or A C, D C; are of the ſame Affection (with reſpect to 180°), as the Sum of the oppoſite Angles A and B, or of A and A D C. Therefore whenever the Sum of the Angles A and B are of the ſame Affection as the Sum of A and the Supplement of B, then tis doubtful whether the other Angle B is acute or obtuſe, becauſe it may be either. But when theſe are of different Affections, that value only of B (whether acute or obtuſe) muſt be taken, which when, added to A, their ſum is of the ſame Affection, as the Sum of the oppoſite Sides. 34.

Cor. 3. And for the ſame Reaſon, when two Angles A and B, and an oppoſite ſide CB are given: If there be two Triangles A C B, D C B that contain the ſame *data*, it will be doubtful whether the other oppoſite Side be AC or its Supplement CD (for CD=CE, the Angles A, D, E being equal) for it may be either of them. Therefore we muſt always take that Value of A C which added to C B, their ſum may be of the ſame Affection (with reſpect to 180), as the Sum of the oppoſite Angles. And if both ſums (*viz.* of AC or its Supplement, added to A) be of the ſame Affection; then it is doubtful. Alſo if CI be⊥AB, it will be ambiguous whether A I or its Sup. E I=D I be the 35.

FIG. Segment of the Baſe, and whether A C I or its Sup.
35. E C I or D C I be the Angle at the Vertex.

Note, the Caſes mentioned in the two laſt Corol. contain the ſix ambiguous Caſes of oblique Triangles. For when two Sides and an Angle oppoſite are given; the other three Things will have different Values, according as an Arch or its Supplement is taken for the other oppoſite Angle. And if two Angles and an oppoſite Side be given; according as an Arch or its Supplement is taken for the other oppoſite Side; the Baſe and its oppoſite Angle will have different Values.

Cor. 4. If in a right angled Triangle, there be given an Angle and its oppoſite Side, there will be two Triangles D B A, C B A that have the ſame *Data*, and therefore the other three Things are doubtful. Whence the Hypothenuſe may be either D B or its Sup. B C. If D B be the Hypothenuſe, then the Baſe is D A and ∠ oppoſite D B A. If B C is the Hyp. then A C is the Baſe and A B C the op. Angle, theſe three being the Sup. of the others. And theſe are called the three ambiguous Caſes in right angled Triangles; when an Angle and its op. Leg is given, to find the reſt.

After the ſame manner in a quadrantal Triangle, if a Side and the oppoſite Angle be given; the Angle oppoſite to the quadrantal Side may be either that Angle D A B, or its Supplement B A C. And therefore the three things ſought are ambiguous, and each of them may be either an Arch or its Supplement.

PROP. XVIII.

In any Triangle A B C. *the ſum of the three Angles is greater than two right Angles.*

30. Let F D E be the ſupplemental Triangle to A B C, by Prop. X. then by Cor. 1. Prop. XIII. $DF+FE+ED$ is leſs than a Circle; that is the Supplements of A, B, and C is $<$ 2 ſemicircles, or $180-A+180-B+$

$B+180-C<360$. ſubtract 2 ſemicircles, and then FIG.
$180-A-B-C<0$ add $A+B+C$. then $180<$ 30.
$A+B+C$, or $A+A+C>180$.

Cor. 1. The ſum of the three Angles of any Triangle is leſs than ſix right Angles. For every Angle muſt be leſs than 2 right Angles, otherwiſe it would be one continu'd Arch.

Cor. 2. In any Triangle D A B, the external Angle 33.
B A C is leſs than the Sum of the interior oppoſite Angles D and B.

Cor. 3. The Sum of any two Angles is greater than the Supplement of the third Angle.

Cor. 4. The Sum of any two Angles—the third (or any Angle and the Difference of the other two) is leſs than two right Angles.

For F D is leſs than F E+E D, by Prop. XIII, that 30.
is $180-B$, is leſs than $180-A+180-C$, that is $180+A+C-B$ is leſs than $180+180$, or $A+C-B$ leſs than 180.

Cor, 5. In a right angled ſpherical Triangle, the Sum of the oblique Angles is greater than one, and leſs than three right Angles. By this Prop. and *Cor*. 4.

PROP. XIX.

If the Angles at the Baſe of a ſpherical Triangle, be 28.
of the ſame Affection, the perpendicular on the Baſe will fall within the Triangle; if of different Affection it falls without.

In the Triangle G A B whoſe Baſe is G B and its Pole P, Since the Perpendicular muſt paſs thro' the Pole P, by Prop. 1. *Cor* 1. let it interſect the Baſe in D, C, then the Angles P G B and P B G are right, and therefore A G B and A B G are both obtuſe, when the Perpendicular C D falls between them. Likewiſe the Angles P F E, and P E F are both right; therefore in the Triangle F A E the Angles F and E are both

FIG. 28. acute, when A C falls within. But in the Triangle B A E the Perpendicular falls without, and the ∠ A B E is acute and A E B obtuſe. And the ſame will hold for any other Triangle that can be drawn.

36. *Cor.* If the two leaſt Sides of a ſpherical Triangle, A C, B C be of the ſame Affection ; the Perpendicular C D upon the other A B, will fall within the Triangle.

Let A B be the greateſt Side, make B E=B C, and A F=A C, then by Cor. 2. Prop. XVI. the ∠ B E C is of the ſame Affection as C B. and A F C of the ſame Affection, as A C; but A C, B C are of the ſame Affection, and therefore the Angles E F C, F E C. Therefore by this Prop. the Perpendicular C D falls within the Triangle E C F or A C B.

If A B be leſs than A C, C B, but greater than their Supplements (they being of the ſame Affection), the Perpendicular will fall within. For this and the adjoining Triangle (whoſe common Baſe is A B) have the ſame Perpendicular, ſee Cor. 2 Prop. IX.

SCHOLIUM

Some other Properties of Perpendiculars may be ſeen in the corollaries to Prop. 8.

PROP. XX.

37. *In any right angled ſpherical Triangle, each of the oblique Angles is of the ſame Affection as the oppoſite ſide.*

If the Angle at A be right, and A C leſs than a Quadrant, produce A C to D, that A D may be a Quadrant, and draw B D; then D is the Pole of B A, and conſequently A B D a right Angle, and therefore A B C is leſs than a right Angle. But if A E be greater than a Quadrant; then its oppoſite Angle A B E is greater than the right Angle A B D. And if A D be a Quadrant, the Angle A B D is a right Angle.

FIG. 37. On the contrary, if A B D is a right Angle, then D is the Pole of A B, and A D a Quadrant; but if A B C be leſs or A B E greater than a right Angle; then the oppoſite Side A C will be leſs, or A E greater than the Quadrant A D.

PROP. XXI.

38. *In a right angled ſpherical Triangle* B A D, *if the Legs be of the ſame Affection, the Hypothenuſe* B A, *will be leſs than a Quadrant; if of different Affection, it will be greater: And the contrary.*

Let D be the right Angle, and produce D B, D A to C; and make D P, D E Quadrants, then P is the Pole of D B, and E of D A, and P B a Quadrant. Then in the Triangle D B A, if D B, D A be leſs than a Quadrant, then B A will be leſs than the Quadrant B P, by Prop. VIII. And in the right angled Triangle B A C, if B C, A C, be greater than a Quadrant; then alſo B A is leſs than BP.

But in the Triangle D B F, where D B is leſs and D F greater than a Quadrant; as likewiſe in the Triangle B C F, the Hypothenuſe B F will be greater than the Quadrant B P. by the ſame Prop. VIII.

On the contrary, if the Hypothenuſe B A be leſs than a Quadrant BP, then DB, DA are both leſs; or elſe C B, C A both greater than a Quadrant. But in the Triangle D B F or C B F, where the Hypothenuſe B F is greater than the Quadrant B P; D B, D F or elſe C B, C F are of different Affections.

Cor. 1. As the two oblique Angles are of the ſame or different Affection; the Hypothenuſe will be accordingly leſſer or greater than a Quadrant: *et contra:* by Prop. 20, and 21.

Cor. 2. As the Hypothenuſe and one Side (or its oppoſite Angle) is of the ſame or different Affection; the other Side (and its oppoſite Angle) will be leſs or greater than a Quadrant.

FIG. 38. *Cor.* 3. As the Hypothenuſe is leſſer or greater than a Quadrant, each Leg will be ſimilar or diſſimilar to its adjacent Angle: and the contrary.

PROP. XXII.

39. *If a Triangle* CIA *have one acute Angle, it alſo hath one Side leſs than Quadrant.*

Let ACI be the acute Angle, and in the Triangle CBA, let C be a right Angle; if either A fall between P and C, or B between Q and C, the Prop. is manifeſt; for then either CA or CB is leſs than a Quadrant. Let therefore CB, CA be greater than Quadrants, then by Prop. 21. BA is leſs than a Quadrant, therefore in the acute angled Triangle ACI, AI is ſtill leſs than a Quadrant.

Cor. If a Triangle have one Side greater than a Quadrant, it alſo hath one obtuſe Angle.

For if a Triangle hath one Side greater than a Quadrant, its ſupplemental Triangle (by Prop. X) hath one acute Angle; and therefore by this Prop. it hath a Side alſo leſs than a Quadrant. And if ſo, then (by Prop. X.) the former has an Angle greater than a right Angle,

PROP. XXIII.

If a Triangle has two Sides leſs than Quadrants, it has alſo an acute Angle.

39. For in the Triangle DBA, if BD, DA be leſs than Quadrants, the Sum of the oppoſite Angles is leſs than 2 right Angles, by Prop. XVII. therefore at leaſt one of the Angles is leſs than a right Angle.

Cor. If a Triangle ACB has two obtuſe Angles, it has alſo one Side greater than a Quadrant.

For by Prop. XVII. If two Angles be greater than two right Angles, then the Sum of their oppoſite Sides is greater than a Semicircle; therefore at leaſt one of them is greater than a Quadrant.

PROP. XXIV.

If the three Angles of a ſpherical Triangle be acute; each Side will be leſs than a Quadrant. 36.

For the Perpendicular C D from any Angle will fall within the Triangle by Prop. XIX. and ſince, in the right angled Triangles A C D, and D C B, the Angles A and A C D, and B and B C D are all acute, therefore by Cor. 1. Prop. XXI. the Hypothenuſes A C, BC are leſs than a Quadrant. And by the ſame Reaſoning A B is leſs than a Quadrant.

Cor 1. If the three Sides of a ſpherical Triangle D E F be greater than Quadrants, the three Angles will be obtuſe. 30.

For then the three Angles A, C, B of the ſupplemental Triangle A C B, will be acute, and therefore the ſides leſs than Quadrants, by this Prop. And therefore by Prop. X. the Angles of the Triangle D E F will be obtuſe.

Cor. 2. In a Triangle D E F. if two Angles D, E be obtuſe, and one F acute; the Sides are of the ſame Affection as the oppoſite Angles.

For in the adjoining Triangle D E T all the Angles are acute by Cor. 2. Prop. IX; and all the Sides leſs than Quadrants by this Prop. and therefore in the Triangle D E F; D F, E F are greater than Quadrants.

Cor. 3. In a Triangle, if two Sides be leſſer, and one greater than a Quadrant; the Angles will be of the ſame Affection as the oppoſite Sides. This follows from Cor 2. of this, and Prop. X.

PROP.

FIG.

PROP. XXV.

39. *In a ſpherical Triangle, when two Sides are of the ſame Affection, and the included Angle acute; then the third Side is leſs than a Quadrant. But if the two Sides are of different Affection, and the included Angle obtuſe; the third Side is greater than a Quadrant.*

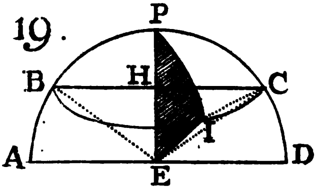
1. In the Triangle B D A or B C A, right angled at D and C, if the Legs are of the ſame Affection, the Hypothenuſe B A is leſs than a Quadrant, by Prop. XXII. but if the Angle at D or C be leſs than 90, as A C I, A D I; then A I is ſtill leſs.

2. Suppoſe the Legs to be of different Affection, then by Prop. XXI. the Hyp. B A will be greater than a Quadrant, and much more will B A be greater than a Quadrant, if the Angle at C and D be obtuſe,

Cor. When two Angles are of the ſame Affection, and the included ſide more than a Quadrant; then the third Angle is obtuſe. But if two Angles are of different Affection, and the included Side leſs than a Quadrant, the third Angle is acute.

For by Prop. X. its ſupplemental Triangle comes under the Caſe of this Prop.

19.



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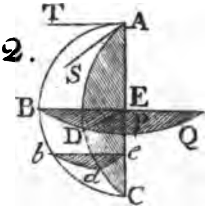

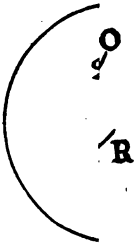

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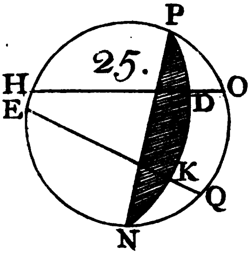

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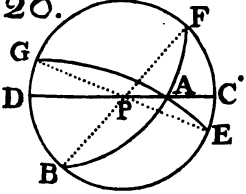

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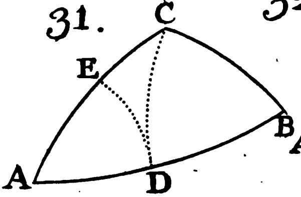

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8.

UNIV. OF MICH.

SECT. III.

The Proportions for calculating the Sides and Angles of ſpherical Triangles,

PROP. XXVI.

In any right angled ſpherical Triangle A B C, *it is,*
As Radius :
To ſine of the Hypothenuſe B C : :
So Sine of any Angle C :
To Sine of its oppoſite Side A B.

FIG. 40.

Let the Triangle B A C be right angled at A; draw AG, C G to the Center G of the Sphere. From B let fall B I ⊥ to A G, and thro' I draw I H ⊥ to G C; and draw B H, which will be in the Plane of the Circle B C. Since the Plane A B G is ⊥ A C G, therefore B I is ⊥ A C G, and C H ⊥ to the Plane B I H. Then B I is the Sine of B A, and B H the Sine of B C, and by Cor. 4. Prop. IV. ∠ B H I = ſpherical Angle A C B.

This done, in the right angled plain Triangle BIH, it will be (by Prop. III. B. II.) Rad : B H : : S.H : B I, that is Rad : S.B C : : S.C : S.A B.

Cor. 1. In right angled Triangles that have equal Angles at the Baſe; the Sines of the Hypothenuſes are as the Sines of the Perpendiculars.

For if ∠C be given, the S.B C is to the S.A B in a given Ratio.

Cor. 2. In right angled Triangles having equal Hypothenuſes; the Sines of the Perpendiculars are as the Sines of their oppoſite Angles. For then B C is given.

Cor.

FIG. 41. *Cor.* 3. In a right angled Triangle ABF, 2 Coſ. Hyp. AF× Rad = Coſ. Sum of the Sides AB, BF + Coſ. their Difference.

For in the Triangle EDF, Rad : S.EF : : S.E : S.DF, that is Rad : Coſ. BF : : Coſ. AB : Coſ. AF, whence Rad × Coſ. AF = Coſ. AB × Coſ. BF = (by Sch. 4. I.) Coſ. $\overline{AB+BF}$ + Coſ. $\overline{AB-BF}$: × into ½ Radius.

PROP. XXVII.

40. *In any right angled Triangle* ABC, *as*
Radius :
Sine of one Side AC : :
Tan. Angle adjacent C :
Tan. oppoſite Side AB.

Suppoſing the ſame Conſtruction as in the laſt Prop. then in the right angled plane Triangle HIG it will be (by Prop. III. B. II.) Rad : S.AGC or AC : : IG : IH. And in the right angled Triangle BIG (right angled at I) Rad : Tan. BGI or AB : : IG : IB. And in the right angled plane Triangle BIH, Tan. BHI or C : Rad : : IB : IH. Therefore *ex equo*, Tan. C : Tan. AB : : IG : IH; that is as Rad : S.AC : : Tan. C : Tan. AB.

Cor. 1. In right angled Triangles having equal Angles at the Baſe; the Sines of the Baſes are as the Tangents of the Perpendiculars.

For S.AC is to Tan. AB in a given Ratio.

Cor. 2. In right angled Triangles having equal Baſes; the Tangents of the Perpendiculars are as the Tangents of the oppoſite Angles.

For if AC be given, theſe are in a given Ratio.

PROP. XXVIII.

FIG. 41.

In any right angled or quadrantal Triangle; the Sine of middle Part and Radius, are reciprocally proportional to the Tangents of the Extreams conjunct, and to the Coſines of the Extreams disjunct.

This in the Lord *Neper*'s Theorem, and is demonſtrated by Induction thus.

Produce the Sides A B, A F, BF to Quadrants, and deſcribe the Arch CDE. Then ſince the Arches AC, AD, BE, CE are Quadrants, the Angles B, C, D will be right; and E D= Complement of D C=Comp. A; and F D=Comp. A F; and E F=Comp. F B. Alſo the Coſine of the Complement is the Sine, and the Cotan. of the Comp. is the Tangent.

Now the middle Part muſt be either a Leg, the Comp. of an Angle, or the Comp. of the Hypothenuſe.

1. Suppoſe a Leg as A B be middle Part, then B F and Comp. ∠ A are Extreams conjunct. But by Prop XXVII. Rad : S. A B : : Tan. A : Tan. F B. Or thus S. A B : Tan. F B : : (Rad : Tan A : :) Cotan. A : Radius, by Cor. 2. Prop. 1. B. I.

2. Let Comp. of an Angle A be middle Part, then A B and Comp. A F are Extreams conjunct; and in the Triangle D E F, by Prop. XXVII. Rad : S.E D : : Tan. E : Tan. D F; or S.E D : Tan. D F : : (Rad : Tan. E : :) Cotan. E : Radius, by Cor. 2. Pr. 1. I. that is Coſ. A : Cotan. A F : : Tan. A B : Rad.

3. Let the Comp. Hypothenuſe A F be middle Part, then Comp. A and Comp. F are Extreams conjunct. Therefore in the Triangle D E F, Rad : S.D F : : Tan. F : Tan. E D; or S.D F : Tan. E D (: : Rad : Tan. F) : : Cotan. F : Radius, that is Coſ. A F : Cotan. A : : Cotan. F : Radius.

Therefore Radius × S. middle Part = Rectangle of the Tangents of the Extreams conjunct. Again

1. Let a Leg A B be middle Part, then Comp. F, and Comp. A F are Extreams disjunct, therefore by Prop. XXVI. Rad : S.A F : : S.F : S.A B.

2. Let

FIG. 41. 2. Let Comp. of an Angle A be middle Part; then F B and Comp. F are Extreams disjunct: Therefore in the Triangle DEF, by Prop. XXVI. Rad : S.EF :: S.F : S.ED, that is Rad : Coſ. BF :: S.F : Coſ. A.

3. Let Comp. Hyp. AF be middle Part; then AB, FB are Extreams disjunct. And in the Triangle DEF, by Prop. XXVI. Rad : S.EF :: S.E : S.DF, that is Rad : Coſ. BF :: Coſ. AB : Coſ. AF.

Therefore Radius × Sine of middle Part = Rectangle of the Coſines of the Extreams disjunct.

And in a quadrantal Triangle, its ſupplemental one, by Prop. X. will be a right angled Triangle. And ſince this Prop. holds in that right angled Triangle, it will equally hold in the quadrantal Triangle, wherein the Parts are the Supplements of the others; becauſe Arches and their Supplements have the ſame Sines, Coſines, Tangents and Cotangents.

Cor. 1. If a Perpendicular FI be let fall on the Baſe of a Spherical Triangle the Coſines of the Angles at the Baſe are as the Sines of the Angles at the Vertex.

For in the right angled Triangles AFI, BFI, it is Coſ. A : S.AFI :: (Coſ. FI : Rad ::) Coſ. FBI : S.BFI.

Cor. 2. The Coſines of the Sides are as the Coſines of the Segments of the Baſe.

For Coſ. AI : Coſ. AF :: (Rad : Coſ. FI ::) Coſ. BI : Coſ. BF.

Cor. 3. The Sines of the Segments of the Baſe are as the Cotangents of the Angles at the Baſe.

For Cotan. A : S.AI :: (Rad : Tan. FI ::) Cot. IBF : S.IB.

Cor. 4. The Cotangents of the Sides, are as the Coſines of the Angles at the Vertex.

For Cotan. A F : Coſ. A F I : : (Rad : Tan. F I : :) Cotan. B F : Coſ. B F I. FIG. 42.

Cor. 5. The Tangents of the Segments of the Baſe are as the Tangents of the Angles at the Vertex.

For Tan. A F I : Tan. A I : : (Rad : S. F I : :) Tan. B F I : Tan. B I.

SCHOLIUM.

This Prop. will reſolve all right angled and quadrantal Triangles; and alſo all oblique ones, (except where 3 ∠'s or 3 ſides are given,) by letting fall a Perpendicular dividing it into two right angled Triangles.

PROP. XXIX.

In any ſpherical Triangle, the Sines of the Sides are proportional to the Sines of their oppoſite Angles. 43.

On A B, A F let fall the Perpendiculars F H, B I; then by Prop. XXVI.

S.A F : S.F H : : *Rad* : S.A
and S.F H : S.F B : : S.B : *Rad*, therefore
ex equo S.A F : S.F B : : S.B : S.A

again S.A B : S.B I : : *Rad* : S.A
and S.B I : S.B F : : S.F : *Rad.* and
ex equo S.A B : S.B F : : S.F : S.A.

Cor. 1. If a great Circle be drawn from the Vertex of a Triangle to cut the Baſe; the Rectangles of of the Sines, of the Sides and of the vertical Angles; are directly as the Sines of the Segments of the Baſe. 44.

For S.C D A : S.C A : : S.C A D : S.C D = $\frac{S.CA \times S.CAD}{S.CDA}$, and S.C D A or B D A : S.A B : : S.D A B : S.D B = $\frac{S.AB \times S.DAB}{S.CDA}$ therefore S.C D : S.D B : : S.C A × S.C A D : S.B A × S.D A B.

Cor.

FIG. 44. *Cor.* 2. If the vertical Angle of a Triangle be biſſected; the Sines of the Segments of the Baſe are as the Sines of the Sides; S.CD : S.BD :: S.AC : S.AB.

PROP. XXX.

42. *In any ſpherical Triangle, if a Perpendicular be let fall on the Baſe, it is*
As the Sine of the Sum of the Sides, AF, BF :
Sine of their Difference ::
Cot. of ½ Sum of the Angles at the Vertex, AFI, BFI :
Tan. of half their Difference.

For by Cor. 4. Pr. XXVIII. Cotan. AF : Cotan. BF :: Coſ. AFI : Coſ. BFI. or by Cor. 4. Pr. I. B. I. Tan. AF : Tan. BF :: Coſ. BFI : Coſ. AFI. And by Compoſition and Diviſion, Tan. AF+Tan. BF : Tan. AF−Tan. BF :: Coſ. BFI+Coſ. AFI : Coſ. BFI−Coſ. AFI, that is by Cor. 4 and 2 Pr. VII. B. I. S.$\overline{AF+BF}$: S.$\overline{AF-BF}$ :: Cotan. $\frac{AFI+BFI}{2}$: Tan. $\frac{AFI-BFI}{2}$.

Cor. 1. If the Perpendicular falls within, $\frac{AFI+BFI}{2}$ = half the Angle at the Vertex F : if it falls without $\frac{AFI-BFI}{2}$ = half the vertical Angle.

Cor. 2. As Sine of Sum of the Sides :
Sine of Difference of the Sides ::
Cotan. half the vertical Angle :
Tan. half Difference, or half Sum, of the vertical Angles, according as the Perpendicular falls within or without.

PROP. XXXI.

42. *Let a Perpendicular fall on the Baſe of a Triangle then*
Tan. half the Sum of the Sides, AF, BF :
Tan. half their Difference ::

Tan. half the Sum of the Angles at the Baſe, B, A : FIG.
Tan. half their Difference. 42.

For by Prop. XXIX. S.A F : S.B F : : S.B : S.A. And by Compoſition and Diviſion, S.A F + S.B F : S.A F — S.B F : : S.B + S.A : S.B — S.A ; that is by Cor. 1. Pr. VII. B. I. Tan. $\frac{AF+BF}{2}$: Tan. $\frac{AF-BF}{2}$: : Tan. $\frac{B+A}{2}$: Tan. $\frac{B-A}{2}$.

PROP. XXXII.

Let a Perpendicular fall on the Baſe of a ſpherical Triangle, then 42.
Cotan. half the Sum of the Angles at the Baſe, A, B :
Tan. half their Difference : :
Tan. $\frac{1}{2}$ *Sum of the Angles at the Vertex*, A F I, B F I :
Tan. half their Difference.

For by Cor. 1. Prop. XXVIII. Coſ.A : Coſ.B : : S.AFI : S.BFI. And Coſ.A + Coſ.B : Coſ.A — Coſ.B : : S.AFI + S.BFI : S.AFI — S.BFI, that is by Cor. 1 and 2. Prop. 7. I. Cotan. $\frac{B+A}{2}$: Tan. $\frac{B-A}{2}$: : Tan. $\frac{AFI+BFI}{2}$: Tan. $\frac{AFI-BFI}{2}$.

Note, If the Perpendicular falls within, $\frac{AFI+BFI}{2}$ = half the vertical Angle F ; but $\frac{AFI-BFI}{2}$, if it fall without.

Cor. Cotan. half the Sum of the Angles at the Baſe :
Tan. half their Difference : :
Tan. half the vertical Angle :
Tan. half the Difference, or half the Sum of the vertical Angles, as the Perpendicular falls within or without.

FIG. 42. This is plain when the Perpendicular falls within; and when it falls without, $+BFI$ becomes $-BFI$, and the contrary.

PROP. XXXIII.

42. *Suppoſe a Perpendicular to fall on the Baſe of a ſpherical Triangle, then as*
Tan. half Sum of the Segments of the Baſe AI, BI :
Tan. half the Sum of the Sides, AF, BF : :
Tan. half Difference of the Sides AF, BF :
Tan. ½ *Diff. of the Segments of the Baſe,* AI, BI.

For by Cor. 2. Prop. XXVIII. Coſ.BF : Coſ.AF : : Coſ.BI : Coſ.AI. And Coſ.BF+Coſ.AF : Coſ.BF—Coſ.AF : : Coſ.BI+Coſ.AI : Coſ.BI—Coſ.AI; that is by Cor. 2. Prop. VII. B. I. Cotan. $\frac{AF+BF}{2}$: Tan. $\frac{AF-BF}{2}$: : Cotan. $\frac{AI+BI}{2}$: Tan. $\frac{AI-BI}{2}$.

Or by Cor. 4. Prop. I. B. I. Tan. $\frac{AI+BI}{2}$: Tan. $\frac{AF+BF}{2}$: : (Cotan. $\frac{AF+BF}{2}$: Cotan. $\frac{AI+BI}{2}$: :) Tan. $\frac{AF-BF}{2}$: Tan. $\frac{AI-BI}{2}$.

Here as the Perpendicular falls within or without, AI+BI or AI—BI will be the Baſe.

Cor. 1. As *Tan.* half the Baſe :
Tan. half Sum of the Sides : :
Tan. half Difference of the Sides :
Tan. half the alternate Baſe.

Where the alternate Baſe is half the Difference or half the Sum of the Segments of the Baſe, according as the Perpendicular falls within or without.

Cor. 2. In a right angled Triangle; the Rectangle of the Tan. of half the Hypothenuſe + half one Leg, and the Tan. half the Hypothenuſe — half that Leg = Tan. Square of half the other Leg, For then BI=0. FIG.

PROP. XXXIV.

Let fall a Perpendicular upon the Baſe of a Triangle, then 42.

Sine of the Sum of the Angles at the Baſe B, A :
Sine of their Difference : :
Tan. half Sum of the Segments of the Baſe, AI, BI :
Tan. half their Difference.

For by Cor. 3. Prop. XXVIII. Cotan. A : Cotan. B : : S.AI : S.BI, or Tan. B : Tan. A : : S.AI : S.BI. And Tan. B+ Tan. A : Tan. B— Tan. A : : S.AI+S.BI ; S.AI—S.BI ; that is by Cor. 4 and 1, Pr. VII. B. I. S.$\overline{B+A}$: S.$\overline{B-A}$: : Tan. $\frac{AI+BI}{2}$: Tan. $\frac{AI-BI}{2}$.

According as the Perpendicular falls within or without, $\frac{AI+BI}{2}$, or $\frac{AI-BI}{2}$ will be equal to half the Baſe. Whence

Cor. 1. *Sine* of the Sum of the Angles at the Baſe :
Sine of their Difference : :
Tan. half the Baſe :
Tan. half the alternate Baſe. (ſee Cor. 1. Pr. 33.)
For when the Perpendicular falls without, +BI becomes —BI, and the contrary.

PROP. XXXV.

Let fall a Perpendicular on the Baſe, then as 42.

Sine of the Sum of the Segments of the Baſe, AI, BI :
Sine of their Difference : :
Sine of the Sum of the Angles at the Vertex, AFI, BFI :
Sine of their Difference.

For

FIG. 42. For by Cor. 5. Prop. XXVIII. Tan. AI : Tan. BI :: Tan. AFI : Tan. BFI. And Tan. AI + Tan. BI : Tan. AI — Tan. BI :: Tan. AFI + Tan. BFI : Tan. AFI — Tan. BFI. that is by Cor. 4. Pr. VII. B. I. $S.\overline{AI+BI} : S.\overline{AI-BI} :: S.\overline{AFI+BFI} : S.\overline{AFI-BFI}$. Here, as the Perpendicular falls within or without; AI+BI, or AI—BI is the Baſe; and AFI+BFI, or AFI—BFI, the vertical Angle. Whence

Cor. As *Sine* of the Baſe :
Sine of the vertical Angle ::
So *Sine* of the diff. Segments of the Baſe :
Sine of the diff. Angles at the Vertex, when the Perpendicular falls within ::
Or *Sine* of the Sum of the Segments of the Baſe :
Sine of the Sum of the vertical Angles, when the Perpendicular falls without.

PROP. XXXVI.

42. *As Coſine of half the Sum of two Sides,* AF, BF :
Coſine of half their Difference ::
Cotan. half the included Angle AFB :
Tan. half Sum of the oppoſite Angles A, ABF.

Put $AF+BF=P$, $AF-BF=p$, $ABF+A=Q$, $ABF-A=q$. $AI+BI=R$, $AI-BI=r$, $AFI+BFI=V$, $AFI-BFI=v$.

Then ſince the Tangents are reciprocally as the Cotangents, we ſhall have by

Prop. 30. $S.P : S.p :: \textit{Cotan.}\,\frac{1}{2}V : T.\frac{1}{2}v$.
31. $\textit{Cot.}\,\frac{1}{2}P : \textit{Cot.}\,\frac{1}{2}p :: T.\frac{1}{2}q : T.\frac{1}{2}Q$
32. $T.\frac{1}{2}q : T.\frac{1}{2}v :: \textit{Cot.}\,\frac{1}{2}V : T.\frac{1}{2}Q$.

And compounding theſe Ratios, and expunging equal Quantities.

$S.P \times \text{Cot.}\,\frac{1}{2}P : S.p \times \text{Cot.}\,\frac{1}{2}p :: \text{Cot. Square, }\frac{1}{2}V :$

Tan.

FIG. 42.

Tan. Square $\frac{1}{2}$Q. But by Sch. Pr. 2. I. $\frac{\overline{2\,\text{Coſ.}^2\,\frac{1}{2}\text{P}}}{\text{Cot.}\,\frac{1}{2}\text{P}} = \text{S.P}$, or $\overline{\text{Coſ.}^2}\,\frac{1}{2}\text{P} = \frac{\text{S.P}\times\text{Cot.}\,\frac{1}{2}\text{P}}{2}$, and $\overline{\text{Coſ.}^2}\,\frac{1}{2}p = \frac{\text{S.}p\times\text{Cot.}\,\frac{1}{2}p}{2}$, therefore Coſ. Square $\frac{1}{2}$P : Coſ. Square $\frac{1}{2}p$ :: Cot. Square $\frac{1}{2}$V : T. Square $\frac{1}{2}$Q; and Coſ. $\frac{1}{2}$P : Coſ. $\frac{1}{2}p$ :: Cot. $\frac{1}{2}$V : Tan. $\frac{1}{2}$Q.

This is plain when the Perpendicular falls within; and when it falls without; inſtead of the third Proportion, ſubſtitute this (by Cor. Pr. XXXII.) T. $\frac{1}{2}q$: T. $\frac{1}{2}$V :: Cot. $\frac{1}{2}v$: T. $\frac{1}{2}$Q. and you will have at laſt, this; Coſ. $\frac{1}{2}$P : Coſ. $\frac{1}{2}p$ :: Cot. $\frac{1}{2}v$: Tan. $\frac{1}{2}$Q.

Cor. 1. As *Sine* of half the Sum of the Sides :
Sine of half their Difference ::
Cotan. half the included Angle :
Tan. half Difference of the oppoſite Angles.

For by the former Proportions, a little vary'd,

S.P : S.p :: Cot. $\frac{1}{2}$V : T. $\frac{1}{2}v$
T. $\frac{1}{2}$P : T. $\frac{1}{2}p$ :: T. $\frac{1}{2}$Q : T. $\frac{1}{2}q$
T. $\frac{1}{2}$Q : T. $\frac{1}{2}v$ :: Cot. $\frac{1}{2}$V : T. $\frac{1}{2}q$

then proceeding as before, it will be found (by Sch. 2. Prop. 2. B. I.) that

S. $\frac{1}{2}$P : S. $\frac{1}{2}p$ :: Cot. $\frac{1}{2}$V : T. $\frac{1}{2}q$. when the Perpendicular falls within; and when it falls without, inſtead of the third Proportion put this (by Cor. Prop. XXXII.) Cot. $\frac{1}{2}$V : Cot. $\frac{1}{2}v$ :: Cot. $\frac{1}{2}$Q : T. $\frac{1}{2}q$. and you will get at laſt S. $\frac{1}{2}$P : S. $\frac{1}{2}p$ :: Cot. $\frac{1}{2}v$: Tan. $\frac{1}{2}q$.

Cor. 2. *Coſ.* half Sum of two Sides :
Coſ. half their Difference ::
Tan. half Difference of their oppoſite Angles :
Tan. half Difference, or half Sum of the vertical Angles; according as the Perpendicular, on the third Side, falls within or without.

This follows from the preſent Prop. and Cor. Prop. XXXII.

FIG. 42.

PROP. XXXVII.

As Coſ. half Sum of two Angles A, ABF :
Coſ. half their Difference ::
Tan. half the included Side AB :
Tan. half Sum of the oppoſite Sides, AF, BF.

Suppoſing as in the laſt Prop. then by

Prop. 31. $\text{Cot.}\,\frac{1}{2}Q : \text{Cot.}\,\frac{1}{2}q :: \text{T.}\,\frac{1}{2}p : \text{T.}\,\frac{1}{2}P$
33. $\text{T.}\,\frac{1}{2}p : \text{T.}\,\frac{1}{2}r :: \text{T.}\,\frac{1}{2}R : \text{T.}\,\frac{1}{2}P$
34. $\text{S.}Q : \text{S.}q :: \text{T.}\,\frac{1}{2}R : \text{T.}\,\frac{1}{2}r$

But (by Sch. Pr. II. B. I.) $\frac{1}{2}\text{S.}Q \times \text{Cot.}\,\frac{1}{2}Q = \overline{\text{Coſ.}}^{2}\,\frac{1}{2}Q$, and $\frac{1}{2}\text{S.}q \times \text{Cot.}\,\frac{1}{2}q = \overline{\text{Coſ.}}^{2}\,\frac{1}{2}q$, therefore by compounding,

$$\overline{\text{Coſ.}}^{2}\,\tfrac{1}{2}Q : \overline{\text{Coſ.}}^{2}\,\tfrac{1}{2}q :: \text{T}^{2}.\,\tfrac{1}{2}R : \text{T}^{2}.\,\tfrac{1}{2}P.$$
$$\text{Or Coſ.}\,\tfrac{1}{2}Q : \text{Coſ.}\,\tfrac{1}{2}q :: \text{T.}\,\tfrac{1}{2}R : \text{T.}\,\tfrac{1}{2}P.$$

But when the Perpendicular falls without, inſtead of the third Proportion, put this (by Cor. Prop. XXXIV), $\text{S.}Q : \text{S.}q :: \text{T.}\,\frac{1}{2}r : \text{T.}\,\frac{1}{2}R$. Whence at laſt you will have this

$$\text{Coſ.}\,\tfrac{1}{2}Q : \text{Coſ.}\,\tfrac{1}{2}q :: \text{T.}\,\tfrac{1}{2}r : \text{T.}\,\tfrac{1}{2}P.$$

Cor. 1. *Sine* half Sum of two Angles, A, ABF:
Sine of half their Difference ::
Tan. half the included Side, AB :
Tan. half Diff. of the oppoſite Sides, AF, BF.

For by varying the Terms of the foregoing Proportions, it will be

$$\text{T.}\,\tfrac{1}{2}Q : \text{T.}\,\tfrac{1}{2}q :: \text{T.}\,\tfrac{1}{2}P : \text{T.}\,\tfrac{1}{2}p.$$
$$\text{T.}\,\tfrac{1}{2}P : \text{T.}\,\tfrac{1}{2}r :: \text{T.}\,\tfrac{1}{2}R : \text{T.}\,\tfrac{1}{2}p.$$
$$\text{S.}\,Q : \text{S.}q :: \text{T.}\,\tfrac{1}{2}R : \text{T.}\,\tfrac{1}{2}r.$$

but (by Sch. Prop. 2. I.) $\frac{1}{2}\text{S.}Q \times \text{T.}\,\frac{1}{2}Q = \text{S}^{2}\,\frac{1}{2}Q$, and $\frac{1}{2}\text{S.}q \times \text{T.}\,\frac{1}{2}q = \text{S}^{2}.\,\frac{1}{2}q$, therefore by compounding,

$$\text{S.}\,\tfrac{1}{2}Q : \text{S.}\,\tfrac{1}{2}q :: \text{T.}\,\tfrac{1}{2}R : \text{T.}\,\tfrac{1}{2}p.$$

If the Perpendicular fall without, inſtead of the third Proportion put this (by Cor. Prop. XXXIV.) S.Q : S.q :: T.$\frac{1}{2}r$: T.$\frac{1}{2}$R. which will give this Proportion, S.$\frac{1}{2}$Q : S.$\frac{1}{2}q$ :: T.$\frac{1}{2}r$: T.$\frac{1}{2}p$. FIG. 42.

Cor. 2. *Coſ*. $\frac{1}{2}$ Sum of the Angles at the Baſe, A, ABF :
Coſ. half their Difference ::
Tan. half Diff. oppoſite Sides, A F, B F :
Tan. half Diff. or half Sum, of the Segments of the Baſe ; according as the Perpendicular falls within or without.

This follows from the preſent Prop. and Cor. 1. Prop. XXXIII.

PROP. XXXVIII.

42.

In any ſpherical Triangle A F B ; *the Rectangle of the Sines of any two Sides* × *Coſ. included Angle* + *the Rectangle of their Coſines* × *Radius* = *Coſ. of the third Side* × *Radius Square ; that is* S.A B × S.F B × *Coſ*. B + *Coſ*. A B × *Coſ*. F B × *Rad* = *Coſ*. A F × *Rad*^{2}.

Let a=S.AB, m=Coſ. AB, s=S.B.
b=S.FB, n=Coſ. FB, c=Coſ. B, v=Cot. B.
r=Rad. $p=\sqrt{rrnn+bbcc}$.

By Prop. 1. I. n : b :: r : Tan. FB :: Cot. FB : r :: (by Prop. XXVIII.) c : Tan. BI $=\frac{cb}{n}$. And by Sch. Pr. 1. I. Sine of BI $=\frac{rcb}{p}$, and Coſ. BI $=\frac{rrn}{p}$; and by Cor. Pr. 6. I. Coſ. AI $=\frac{abc+mnr}{p}$, but by Cor. 2. Prop. XXVIII. $\frac{rrn}{p}:\frac{abc+mnr}{p}::n$: Coſ. AF $=\frac{abc+mnr}{rr}$

Cor. 1. The Coſine of any Angle of a ſpherical Triangle is equal to the Coſine of the oppoſite Side × Radius Square — the Rectangle of the Coſines of the includ-

FIG. 42. including Sides × Radius, and the whole divided by the Rectangle of the Sines of the including Sides.

For $\text{Coſ. B} = \dfrac{\text{Coſ. AF}\times rr - \text{Coſ. AB}\times\text{Coſ. FB}\times r}{\text{S.AB}\times\text{S.FB}}$.

Cor. 2. Cotan. A =

$$\frac{\text{S.AB}\times\text{Coſ. FB}\times\text{Rad.} - \text{S.FB}\times\text{Coſ. AB}\times\text{Coſ. B}}{\text{S.B}\times\text{S.FB}}.$$

For by Prop. 6. I. $\text{S.AI} = \dfrac{anr - bcm}{p}$, and by Cor. 3. Prop. XXVIII. $\dfrac{rcb}{p} : \dfrac{anr - bcm}{p} :: \tau : \text{Cot. A} = \dfrac{anr - bcm}{rcb}\tau = \dfrac{anr - mbc}{bs}$, becauſe $\dfrac{\tau}{c} = \dfrac{r}{s}$ by Sch. Pr. 1. I.

Cor. 3. $\dfrac{\text{S.}\overline{\text{AB} - \text{FB}}\times rr + \text{S.FB}\times\text{Coſ. AB}\times\text{verſ. B}}{\text{S.B}\times\text{S.FB}}$ $= \text{Cotan. A}$.

For ſince $\text{Coſ. B} = r - \text{verſ. B}$, therefore by Cor. 2. $\text{Cot. A} = \dfrac{anr - mbr + mb\times\text{verſ. B}}{bs}$; but by Prop. 6. I. $an - bm = \text{S.}\overline{\text{AB} - \text{FB}}\times r$, whence the Cor. is evident.

SCHOLIUM.

If the Angle ABF be greater than a right Angle, then its Coſine c will be negative.

PROP. XXXIX.

45. *The Rectangle of the Sines of two Angles × Coſine of the included Side divided by Radius Square — the Rectangle of their Coſines divided by Radius, is equal to the Coſine of the third Angle.*

that is, $\dfrac{\text{S.A}\times\text{S.B}\times\textit{Coſ.}\,\text{AB}}{rr} - \dfrac{\textit{Coſ.}\,\text{A}\times\textit{Coſ.}\,\text{B}}{r} = \textit{Coſ.}\,\text{F}$.

Let fall the Perpendicular BR, and put FIG. 45.

S.A$=a$, Coſ.A$=m$, Cot.A$=\tau$, Coſ.AB$=c$, Rad$=r$.
S.B$=b$, Coſ. B$=n$, $q=\sqrt{\tau\tau+cc}$

then by Prop. XXVIII. Cot. ABR$=\frac{rc}{\tau}$, and (by Sch. 1. I.) its Sine $=\frac{r\tau}{q}$, and its Coſine $=\frac{rc}{q}$. And by Prop. 6. I. S.RBF$=\frac{bc-n\tau}{q}$; and by Cor. 1. Pr. XXVIII. $\frac{r\tau}{q} : \frac{bc-n\tau}{q} :: m :$ Coſ. F$=\frac{bc-n\tau}{r\tau}m=\frac{bcm}{r\tau}-\frac{mn}{r}=\frac{abc}{rr}-\frac{mn}{r}$; becauſe $\frac{m}{\tau}=\frac{a}{r}$, by Sch. 1. I.

Cor. 1. The *Coſine* of any Side is equal to the Rectangle of the Coſines of the including Angles × *Radius* + *Coſ.* oppoſite Angle × *Radius* ſquare, and the whole divided by the Rectangle of the *Sines* of the including Angles.

For c or the *Coſ.* AB$=\frac{mnr+rr\times\text{Coſ. F}}{ab}$.

Cor. 2. *Cot.* FB$=\frac{\text{S.B}\times\text{Cot. A}}{\text{S.AB}}+\frac{\text{Coſ. B}\times\text{Cot. AB}}{r}$.

For let $s=$S.AB, $t=$Cotan. AB, then by Cor. 1. Prop. 6. I. Coſ. RBF$=\frac{b\tau+nc}{q}$, and by Cor. 4. Prop. XXVIII. $\frac{rc}{q} : \frac{b\tau+nc}{q} :: t :$ Cot. BF$=\frac{b\tau+nc}{rc}t=\frac{b\tau}{s}+\frac{nt}{r}$, becauſe $ts=cr$. Or Cot. BF$=\frac{rbm+nac}{sa}$, becauſe $\tau=\frac{mr}{a}$, and $t=\frac{cr}{s}$, by Sch. Prop. 1. I.

Cor. 3. Cotan. FB =

$$\frac{r^2\times \text{S.}\overline{\text{A}+\text{B}}-\text{S.A}\times\text{Coſ.B}\times\text{verſ.AB}}{\text{S.A}\times\text{S.AB}}.$$

FIG. For let $v=$ verſ. AB $=r-c$, and $c=r-v$, then Cot.
45. FB $=\frac{rbm+nac}{sa}=\frac{rbm+ran-anv}{sa}$; but (by Pr. 5. I.) $mb+an=r\times S.\overline{A+B}$, whence Cot. BF $=\frac{rr\times S.\overline{A+B}-anv}{as}$.

PROP. XL.

Let a, s, b, p, f, d be the Sines, and m, c, n, i, g, e the Coſines $\Big\}$ *of the Angles and*
47. *Sides, as mark'd in the Figure. And*
$bb=bb-aa=mm-nn$.
and $kk=1-bbpp=1-aadd$.
Radius $=1$. Then

Sine of $\angle$ F $=\frac{bb}{bim-aen}=\frac{bim+aen}{kk}$.

For 1. $bif-ng=m$. By Prop. XXXIX.
2. $aef-mg=n$. By Prop. XXXIX.
3. $\frac{n}{b}=\frac{pe-dig}{df}$. By Cor. 2. Pr. XXXVIII.

For by Pr. 1. I. $\frac{b}{n}=$ Cotan. B. From the two firſt Equations $g=\frac{bif-m}{n}=\frac{aef-n}{m}$, whence $bifm-m^2=aefn-nn$, and $f=\frac{m^2-n^2}{bim-aen}$.

Again, by the third Equation $dnf=bpe-bdig=ade-bdig$, becauſe (by Prop. XXIX.) $bp=ad$; and $nf=ae-big$; whence $g=\frac{ae-nf}{bi}=\frac{bif-m}{n}$, and $aen-fn^2=b^2i^2f-bim$, whence $f=\frac{aen+bim}{n^2+bbii}=\frac{aen+bim}{1-bbpp}$; becauſe $ii=1-pp$.

Cor. 1. Coſ. F $=\frac{abei-mn}{kk}=\frac{aem-bin}{bim-aen}$.

For by the 1st and 3d Equations $f=\frac{ng+m}{bi}=\frac{ae-big}{n}$, FIG. 47. and $nng+mn=aebi-bbiig$; whence $g=\frac{aebi-mn}{nn+bbii}=\frac{aebi-mn}{1-bbpp}$.

Again, by the 1st and 2d Equations $f=\frac{m+ng}{bi}=\frac{n+mg}{ae}$, and $aem+aeng=bin+bimg$, whence $g=\frac{aem-bin}{bim-aen}$.

Cor. 2. Cotan. $F=\frac{ame-bin}{bb}=\frac{e\times S.2A-i\times S.2B}{\text{Coſ. }2A-\text{Coſ. }2B}$

For multiplying the 1st Equation by n, and the 2d by m, we have $mn=binf-nng=aefm-mmg$, and $\overline{m^2-n^2}.g=\overline{aem-bin}.f$, and $\frac{g}{f}=\frac{aem-bin}{m^2-n^2}=\frac{2aem-2bin}{2b^2-2a^2}$; but (by Prop. 2. I.) $2am=$ S.2A, and $2bn=$ S.2B, alſo $2aa=$ verſ. 2 A, and $2b^2=$ verſ. 2B, and $2bb-2aa=$ verſ. 2B — verſ. 2 A = Coſ. 2 A — Coſ. 2 B, and $b^2-a^2=m^2-n^2$, for $bb+n^2=1=m^2+a^2$.

SCHOLIUM.

Any of the Quantities a, b, p, d, or m, n, i, e, may be expung'd by help of the Equation $bp=ad$, or the Tangents or Cotangents ſubſtituted for them.

PROP. XLI.

Suppoſe as in the laſt Prop. and let 47.

$ll=dd-pp=ii-ee$,
$tt=1-bbpp=1-aadd$
Radius $=1$. *Then*
Sine of AB $=\frac{ll}{mid-pen}=\frac{mid+pen}{tt}$.

FIG. 47.

For 1. $pns+ci=e$, by Prop. XXXVIII.
2. $mds+ce=i$, by Prop. XXXVIII.
3. $\frac{i}{p}=\frac{bm+anc}{as}$, by Cor. 2. Prop. XXXIX.

By the two firſt Equations $c=\frac{e-pns}{i}=\frac{i-mds}{e}$, and $ee-pens=ii-imds$, and $imds-pens=ii-ee$, whence $s=\frac{ii-ee}{dim-pen}$.

Alſo by the third Equation $asi=pbm+pnac=$ (ſince $ad=bp$) $adm+pnac$, that is $si=dm+pnc$, whence $c=\frac{si-dm}{pn}=\frac{e-pns}{i}$, and $si^2-dim=pen-p^2n^2s$, whence $s=\frac{pen+dim}{i^2+p^2n^2}=\frac{pen+dim}{1-ppbb}$.

Cor. 1. *Coſ.* AB $=\frac{ei-mndp}{tt}=\frac{pin-dem}{pen-dim}$.

For by 1st and 3d Equations, $s=\frac{e-ci}{pn}=\frac{dm+cpn}{i}$, and $ei-ci^2=pndm+p^2n^2c$, whence $c=\frac{ei-pdmn}{i^2+p^2n^2}=\frac{ei-pdmn}{1-bbpp}$.

Alſo by the 1st and 2d Equations $s=\frac{e-ci}{pn}=\frac{i-ce}{md}$, and $mde-mdci=pni-pnec$, whence $c=\frac{dem-pin}{dim-pen}$.

Cor. 2. *Cotan.* AB $=\frac{med-pin}{ll}=$
$\frac{m\times \text{S.}2\,\text{AF}-n\times \text{S.}2\,\text{BF}}{\text{Coſ.}\,2\,\text{BF}-\text{Coſ.}\,2\,\text{AF}}$.

For multiplying the 1st Equation by i, and the 2d by e, we have $pnsi+cii=ei=mdse+cee$, and $cii-cee=mdes-pins$, whence $\frac{c}{s}=\frac{med-pin}{ii-ee}=\frac{2med-2pin}{2dd-2pp}$; but by Prop. 2. I. $2ed=$ S.2AF, and $2pi=$ S.2BF, and $2dd=$ verſ. 2AF, and $2p^2=$ verſ. 2BF, and $2dd-2pp=$ verſ. 2AF — verſ. 2BF = Coſ. 2BF — Coſ. 2AF.

SCHO-

FIG. 47.

SCHOLIUM.

Any of the Quanties a, d, b, p; or m, e, i, n, may be exterminated by help of the Equation $ad=bp$, or the Tangents or Cotangents ſubſtituted inſtead of them.

PROP. XLII.

In any ſpherical Triangle, ABF,
As Rectangle of the Sines of the Sides AB, FB : 46.
Radius Square : :
Verſ. Baſe — verſ. difference of the Sides :
Verſ. included Angle B : :
Coſ. difference of the Sides — Coſ. Baſe :
Verſ. included Angle B.

Let $a=$S.AB, $m=$Coſ. AB, $c=$Coſ. B. $v=$verſ. B.
$b=$S.FB, $n=$Coſ. FB, $d=$Coſ.AF, Rad $=r$.

Then (by Cor. 2. Prop. XXXVIII.) $\frac{dr^2-mnr}{ab}=c=r-v$, and $v=\frac{abr-drr+mnr}{ab}$; but (by Cor. 1. Prop. 6. I.) $\frac{ab+mn}{r}=r\times$ Coſ. $\overline{\text{AB}-\text{FB}}$. Therefore

$$v=\frac{rr\times\text{Coſ.}\,\overline{\text{AB}-\text{FB}}-drr}{ab}=\frac{\text{verſ.AF}-\text{verſ.}\,\overline{\text{AB}-\text{FB}}}{ab}\times rr.$$

Cor. 1. Rectangle of the Sines of the Sides :
S. half Sum×S. half diff. of the Baſe and diff. Sides : :
And *verſ.* Sum of the Sides—*verſ.* their difference :
verſ. Baſe — verſ. difference of the Sides : :
And *Coſ.* difference — *Coſ.* Sum of the Sides :
Coſ. diff. Sides — *Coſ.* Baſe : :
So twice *Radius* :
verſ. Angle included, B.

For let the greater Side AB$=a$, leſſer BF$=e$, Baſe AF$=b$, $a+e=z$, $a-e=d$, $v=$ verſ. B.

Then

FIG. 46. Then by this Prop. S.a×S.e : $\overline{\text{verſ. }b-\text{verſ. }d}\times r$ :: $r : v$. But (by Cor. 3. Pr. 3. I.) $\frac{1}{2}r\times\overline{\text{verſ. }b-\text{verſ. }d}$ $=$S.$\frac{b+d}{2}$×S.$\frac{b-d}{2}$, and (by Cor. 2, Pr. 3. I.) S.a×S.e $=\frac{1}{2}r\times\overline{\text{Coſ. }d-\text{Coſ. }z}$. Whence you have theſe two Proportions,

$$\text{S.}a\times\text{S.}e : \text{S.}\frac{b+d}{2}\times\text{S.}\frac{b-d}{2} :: 2r : v.$$

And verſ. z—verſ. d : verſ. b—verſ. d :: $2r : v$.
And the Diff. of the verſed Sines is = diff. Coſines.

Cor. 2. Rectangle of the *Sines* of the Sides :
Radius Square : :
Verſ. Sum of the Sides — *Verſ.* Baſe.
Verſ. Sup. of the included *Angle* B.
For (by Cor. 1.)
Verſ. z—verſ. d : verſ. b—verſ. d :: $2r : v$.
And by Diviſion
Verſ. z— verſ. d : verſ. z— verſ. b :: $2r$: $2r-v$,
Or Verſ. z— verſ. d : $2r$ :: verſ. z— verſ. b : $2r-v$.
And $\frac{\text{verſ.}z-\text{verſ.}d}{2}\times r : rr$ :: verſ. z—verſ. b : $2r-v$.
that is (by Cor. 2. Prop. 3. I.) S.a × S.e : rr :: verſ. z —verſ. b : $2r-v$, the verſ. Sup. B. and the diff. verſed Sines = diff. Coſines. Whence

Cor. 3. *Verſ.* Sum of the Sides — *verſ.* diff. Sides :
Verſ. Sum Sides — *verſ.* Baſe : :
Twice *Radius* :
Verſ. Supplement of the included Angle B.

Cor. 4. As Rectangle of the *Sines* of the Sides :
Radius Square : :
Or as Sum of the *Sines* × diff. *Sines*, of ½ *Sum* and ½ *diff.*
Radius Square : : [Sides :
So S.$\overline{\frac{1}{2}\text{ Baſe}+\frac{1}{2}\text{ diff. Sides}}$ × S.$\overline{\frac{1}{2}\text{ Baſe}-\frac{1}{2}\text{ diff. Sides}}$:
Sine Square of half the included Angle : :
So S. ½ Sum of the 3 Sides × S.$\overline{\frac{1}{2}\text{ Sum 3 Sides}-\text{Baſe}}$:
Coſine Square of half the included *Angle* B.

Let $V=\text{Sup. }B$, $s=\text{S.}\frac{1}{2}B$, $c=\text{Coſ.}\frac{1}{2}B$. The reſt as before, then (by Pr. 2. I.) $v=\frac{2ss}{r}$, and $V=\frac{2cc}{r}$, and (by Prop. 4. I.) $\frac{\text{verſ.}\,b-\text{verſ.}\,d}{2}\times r=\text{S.}\frac{b+d}{2}\times\text{S.}\frac{b-d}{2}$, and $\frac{\text{verſ.}\,z-\text{verſ.}\,b}{2}\times r=\text{S.}\frac{z+b}{2}\times\text{S.}\frac{z-b}{2}$; and (by Cor. 2. Prop. 4. I.) $\text{S.}a\times\text{S.}e=\overline{\text{S.}\frac{a+e}{2}+\text{S.}\frac{a-e}{2}}\times\overline{\text{S.}\frac{a+e}{2}-\text{S.}\frac{a-e}{2}}=\overline{\text{S.}\frac{1}{2}z+\text{S.}\frac{1}{2}d}\times\overline{\text{S.}\frac{1}{2}z-\text{S.}\frac{1}{2}d}$. Now by this Prop. $\text{S.}a\times\text{S.}e : rr :: \text{verſ.}\,b-\text{verſ.}\,d : v$ or $\frac{2ss}{r} :: \frac{\text{verſ.}\,b-\text{verſ.}\,d}{2}\times r$ or $\text{S.}\frac{b+d}{2}\times\text{S.}\frac{b-d}{2} : ss$.

FIG. 46.

Again $a+e+b=z+b$, and $\frac{a+e+b}{2}-b=\frac{z-b}{2}$, and by Cor. 2. $\text{S.}a\times\text{S.}e : rr :: \text{verſ.}\,z-\text{verſ.}\,b : V$ or $\frac{2cc}{r} :: \frac{\text{verſ.}\,z-\text{verſ.}\,b}{2}\times r$ or $\text{S.}\frac{z+b}{2}\times\text{S.}\frac{z-b}{2} : cc$. Hence alſo, ſince $d=a-e$,

Cor. 5. As Rectangle of the *Sines* of the Sides :
Radius Square ::
So $\text{S.}\frac{1}{2}$ Sum of 3 Sides—one Side, × $\text{S.}\frac{1}{2}$ Sum — other Side :
Sine Square of half the included *Angle*.

Cor. 6. From half the Sum of the 3 Sides, ſubtract the Baſe, and two Sides ſeparately; then
As S. half Sum of the 3 Sides × S. firſt Difference :
Rectangle of the *Sines* of the 2^d and 3^d Differences ::
So *Radius* Square :
Tan. Square of half the included Angle.

For by Cor. 4. $\text{S.}\frac{z+b}{2}\times\text{S.}\frac{z-b}{2} : \text{S.}\frac{b+d}{2}\times\text{S.}\frac{b-d}{2} :: (cc : ss ::)\ rr$: Tan. Square of half the Angle, by Prop. 1. I.

SCHO-

FIG. 46.

SCHOLIUM.

Any of theſe compound Proportions may eaſily be reſolved into two ſimple ones. For Example in Cor. 4. ſay S.a : r^2 : : S.$\frac{b+d}{2}$: to the Sine of a fourth A. Then S.e : S.A : S.$\frac{b-d}{2}$: ss, required. And the ſame may be done for the next Prop. and its Corollaries.

PROP. XLIII.

42 *In any ſpherical Triangle,*
As Rectangle of the Sines of two Angles A, B :
Radius Square : :
Verſ. Sum of the including Angles—verſ. Sup. 3^d *Angle* :
Verſ. included Side : :
And ſo verſ third Angle—verſ. Sup. Sum of the including Angles :
Verſ. included Side : :
And ſo Coſ. diff. of the including Angles + *Coſ.* 3^d *Angle* :
Verſ. Sup. included Side : :
And ſo verſ. Sup. 3^d *Angle — verſ. diff. including Angles*:
Verſ. Sup. included Side : :
And ſo verſ. Sup. diff. including Angles—verſ. 3^d *Angle* :
Verſ. Sup. included Side, A B.

For let B be the greater Angle, A the leſſer, A B the included Side, and put

a=S.A, m=Coſ.A, c=Coſ. AB, v=verſ. AB,
b=S.B, n=Coſ.B, d=Coſ.F, V=verſ. Sup. A B,

then (by Cor. 1. Pr. XXXIX.) $\frac{mnr+drr}{ab}=c=r-v$, therefore $v=\frac{abr-mnr-drr}{ab}$; but (by Cor. 1. Pr. 5. I.) $\frac{mn-ab}{r}=$ Coſ. $\overline{A+B}=$ (by Sch. 1. I.) $r-$

verſ.

FIG. 42.

verſ. $\overline{A+B}=$ verſ. Sup. $\overline{A+B}-r$. and $d=r-$ verſ. F $=$ verſ. Sup. F $-r$, whence $v=\frac{\text{verſ. }\overline{A+B}-\text{verſ. Sup. F}}{ab}rr$

$=\frac{\text{verſ. F}-\text{verſ. Sup. }\overline{A+B}}{ab}rr.$

Again $V=r+c=\frac{abr+mnr+drr}{ab}$. But (by Cor. 1. Pr. 6. I.) $\frac{ab+mn}{r}=$ Coſ. $\overline{B-A}=r-$ verſ. $\overline{B-A}=$ verſ. Sup. $\overline{B-A}-r$. Therefore $V=\frac{\text{Coſ. }\overline{B-A}+\text{Coſ. F}}{ab}\times rr=$

$\frac{\text{verſ. Sup. F}-\text{verſ. }\overline{B-A}}{ab}rr=$

$\frac{\text{verſ. Sup. }\overline{B-A}-\text{verſ. F}}{ab}rr.$

Cor. 1. As *verſ.* Sum of two Angles — *verſ.* Diff.:
To twice *Radius* ::
Or as *Coſ.* Difference of two Angles — *Coſ.* Sum :
To twice *Radius* ::
So *verſ.* Sum of the including Angles — *verſ.* *Sup.* third *Angle* :
Verſ. included Side ::
So *Verſ.* third Angle — *Verſ.* *Sup.* Sum of the including *Angles* :
verſ. included Side ::
And ſo *Coſ.* Diff. including Angles + *Coſ.* 3^d Angle :
Verſ. *Sup.* included Side ::
And ſo *verſ.* *Sup.* 3^d Angle — *verſ.* Diff. including Angles :
Verſ. *Sup.* included Side ::
And ſo *verſ.* *Sup.* Diff. incl. Angles — *verſ.* 3^d Angle :
Verſ. *Sup.* included Side, A B.

For (by Cor. 2. Pr. 3. I.) $S.A\times S.B=$

$\frac{\text{Coſ. }\overline{B-A}-\text{Coſ. }\overline{B+A}}{2}\times r=$

$\frac{\text{verſ. }\overline{B+A}-\text{verſ. }\overline{B-A}}{2}\times r$. And all the reſt follows from this Prop.

FIG. 42. *Cor.* 2. As Rectangle of the *Sines* of two Angles A, B :
Radius Square : :
So S. $\frac{1}{2}$ Sum of the including $\angle$'*s* + $\frac{1}{2}$ Sup. $3^d \angle \times$
S. $\frac{1}{2}$ Sum including $\angle$ *s* — $\frac{1}{2}$ Sup. $3^d \angle$:
Sine Square of half the included Side : :
And ſo S. $\frac{1}{2}$ Sum $\overline{\text{of } 3^d \angle, \text{ and Sup. Sum incl.}}$ $\angle$'s $\times$
S. $\frac{1}{2}$. their Diff. :
Sine Square of half the included Side : :
So S. $\frac{1}{2}$ Sup. $3^d \angle + \frac{1}{2}$ Diff. incl. $\angle$'s $\times$ S. $\frac{1}{2}$ Sup. $3^d \angle$ —
$\frac{1}{2}$ Diff. incl. $\angle$'s :
Coſ. Square of half the included Side : :
So S. $\frac{1}{2}$ Sup. Diff. incl. $\angle$ *s* + $\frac{1}{2}$ the $3^d \angle \times$ S. $\frac{1}{2}$ Sup.
Diff. incl. $\angle$*s* — $\frac{1}{2}$ the $3^d \angle$:
Coſ. Square of half the included Side, A B.

For ſince the Diff. Coſ. = Diff. verſed Sines, therefore (by Cor. 1. Pr. 3. I.) $\frac{\text{verſ. } \overline{B+A} - \text{verſ. Sup. } F}{2} \times r$ $= S.\ \frac{B+A+\text{Sup. } F}{2} \times S.\ \frac{B+A-\text{Sup. } F}{2}$.

And $\frac{\text{verſ. } F - \text{verſ. Sup. } \overline{B+A}}{2} \times r =$ $S.\ \frac{F+\text{Sup. } \overline{B+A}}{2} \times S.\ \frac{F-\text{Sup. } \overline{B+A}}{2}$.

And $\frac{\text{verſ. Sup. } F - \text{verſ. } \overline{B-A}}{2} \times r =$ $S.\ \frac{\text{Sup. } F+B-A}{2} \times S.\ \frac{\text{Sup. } F+A-B}{2}$.

And $\frac{\text{verſ. Sup. } \overline{B-A} - \text{verſ. } F}{2} \times r =$ $S.\ \frac{\text{Sup. } \overline{B-A}+F}{2} \times S.\ \frac{\text{Sup. } \overline{B-A}-F}{2}$.

And (by Pr. 2. I.) verſ. A B $\times \frac{1}{2} r =$ Sine Square of $\frac{1}{2}$ A B, and verſ. Sup. A B $\times \frac{1}{2} r =$ Coſ. Square of $\frac{1}{2}$ A B, then all the reſt follows from the preſent Prop.

Cor.

FIG. 42.

Cor. 3. Rectangle of the *Sines* of two Angles A, B :
Radius Square : :
Or Sum of the *Sines* × Diff. *Sines*, of ½ Sum and ½ Diff. [Angles :
Radius Square : :
So *Coſ.* ½ Sum × *Coſ.* ½ Diff. of the 3^d ∠ and Sum incl. [Angles :
Sine Square of half the included Side : :
So *Coſ.* ½ Sum × *Coſ.* ½ Diff. of 3^d ∠, and Diff. incl. [Angles :
Coſ. Square of half the included Side, A B.

For (by Cor. 2. Pr. 4. I.) $\text{S.}A\times\text{S.}B=\text{S.}\frac{B+A}{2}+\text{S.}\frac{B-A}{2}\times\text{S.}\frac{B+A}{2}-\text{S.}\frac{B-A}{2}$; and $\text{S.}\frac{B+A+180-F}{2}\times\text{S.}\frac{B+A-180+F}{2}=\text{S.}90+\frac{B+A-F}{2}\times\text{S.}\frac{B+A+F}{2}-90=\text{Coſ.}\frac{F-A-B}{2}\times\text{Coſ.}\,180-\frac{B+A+F}{2}=\text{Coſ.}\frac{F-A-B}{2}\times\text{Coſ.}\frac{F+A+B}{2}$.

Again $\text{S.}\frac{180-F+B-A}{2}\times\text{S.}\frac{180-F+A-B}{2}=\text{S.}90-\frac{F+A-B}{2}\times\text{S.}\,90-\frac{F+B-A}{2}=\text{Coſ.}\frac{F-B+A}{2}\times\text{Coſ.}\frac{F+B-A}{2}$. The reſt follows from Cor. 2.

Cor. 4. From half the Sum of the three Angles, ſubtract each Angle at the Baſe ſeparately, and then the third Angle. Then
As Rect. of the *Coſines* of the 2 firſt Differences :
Rect. of the *Coſ.* 3^d Diff. and *Coſ.* ½ ſum of 3 Angles : :
So *Radius* Square :
Tan. Square of half the included Side or Baſe, A B.
This follows from Cor. 3. becauſe Coſ. : Sine : : Rad : Tangent. See Schol. Prop. XLII.

FIG.

PROP. XLIV.

48. *If the vertical Angle* F *of a Triangle* A F B *be biſected, it will be*
As twice the Cot. of the biſecting Line F I :
Sum of the Cotangents of the Sides A F, B F : :
Radius :
Coſ. half the vertical Angle F.

Produce FA, F I, F B to Quadrants, and to the Pole F draw the great Circle LT R, and produce B A to interſect it in R. And let Tan. A S$=b$, Tan. I T$=c$, Tan. B L$=d$, S.RT$=y$, Coſ. $\frac{1}{2}$ F$=x$. Then the Angles at S, T, L are right. And by Cor. I. Prop. XXVII. $c : b :: y : \frac{by}{c} =$ S.R S. And $c : d :: y : \frac{dy}{c} =$ S.R L. And ſince S T=T L; therefore (by Prop. 3. I.) $\frac{by+dy}{c} r = 2x \times y$, or $\frac{b+d}{c} \times r = 2x$, that is $2cx = \overline{b+d} \times r$.

Cor. As twice *Radius* :
Tan. biſecting Arch : :
Sum of the *Cotan.* Sides :
Coſ. half the vertical Angle F.

PROP. XLV.

48. *If an Arch be drawn from the Vertex to the Middle of the Baſe of a Triangle; then the Sines of the vertical Angles are reciprocally as the Sines of the Sides, and directly as the Sines of the Angles at the Baſe.*

For (by Cor. 1. Pr. XXIX.) S.A F I × S.A F = S.B F I × S.B F, becauſe A I=B I. Therefore S.AFI : S.B F I : : S.B F : S.A F : : (by Pr. XXIX.) S.A : S.B.

PROP. XLVI.

FIG. 49.

As one right Angle:
To the Angle intercepted between two great Circles::
So the Area of a great Circle of the Sphere:
Lunular Area containd between theſe great Circles.

Let A B C, A D C be the two great Circles; B D F a great Circle whoſe Poles are A and C. Divide the Circle B D F into an infinite Number of equal Parts, thro' all which draw great Circles paſſing thro' A and C, which will divide the Surface of the Sphere into the ſame Number of equal Parts, ſimilar and equal to one another; ſince their Baſes (in the Circle BF), and Sides are all equal. Therefore as the Number of Parts in the whole Circumference, to the Number of Parts in B D, that is, as the whole Circumference, to the Arch B D : : ſo the Surface of the Sphere to the Area A B C D A; or as 4 right Angles: Angle B A D : : Surface of the Sphere: Area A B C D A. But (by Geometry) the Surface of the Sphere = 4 great Circles. Therefore as 4 right Angles: Angle B A D : : 4 great Circles: Area A B C D A; or as one right Angle : one great Circle : : Angle B A D: Area A B C D A.

Cor. As 4 right Angles:
Intercepted Angle B A D : :
So Surface of the Sphere:
Surface intercepted between theſe two great Circles::
And ſo Solidity of the Sphere:
Solidity contain'd between theſe great Circles.

For the Solidities are as the Surfaces, becauſe they may be reſolved into Pyramaids of equal height.

PROP. XLVII.

In any Spherical Triangle, G; 50.
As two right Angles:
Exceſs of the three Angles above two right Angles::
Area of a great Circle of the Sphere:
Area of the Triangle.

FIG. 59. Produce all the Sides to Semicircles, then the oppoſite Angles are equal (by Cor. 1. Prop. IV.); and (by Prop. IX) Triangle H = Triangle G, alſo half the Surface of the Sphere = G + R + S + T; let a, b, c be the three Angles, then (by Cor. Prop. XLVI.)

180 : a :: $\frac{1}{2}$ Surface Sphere : G + R
180 : b :: $\frac{1}{2}$ Surface Sphere : G + S
180 : c :: $\frac{1}{2}$ Surface Sphere : H + T or G + T,

therefore 180 : $a+b+c$ :: $\frac{1}{2}$ Surface Sphere : 3G + R + S + T, and by Diviſion, 180 : $a+b+c-180$ :: $\frac{1}{2}$ Surface Sphere : 2G :: $\frac{1}{4}$ Surface Sphere, or the Area of a great Circle : G.

Cor. 1. As 1 : 57.2957795 :: the 3 Angles — 2 right Angles : Area, in ſquare Degrees.

For Area of a great Circle = $\frac{1}{2}$ Circumference × Radius = 180 × Radius, and Radius = 57.2957795 *&c.*

Cor. 2. As 180 : 3 Angles — 180 :: $3.1416 \times \overline{\text{Rad}}^2$: Area of the Triangle.

For $3.1416\,rr$ = Area of the Circle, whoſe Radius is r.

Cor. 3. In any ſpherical Polygon, put n = Number of Sides, $A = 180° \times \overline{n-2}$. Then I ſay

As 2 right Angles or 180° :
Sum of all the Angles of the Polygon — A ::
So the Area of a great Circle of the Sphere :
Area of the Polygon.

Suppoſe the Polygon to be divided into as many Triangles as it has Sides, by Arches drawn from a Point within it. Then Since (by this Prop.) it is in any one Triangle, as 180 : a great Circle :: Sum of its Angles — 180 : its Area; therefore by Compoſition it will be, 180 : a great Circle :: Sum of all the Angles internal and external — $n \times 180$: Sum of all the Areas of the Triangles; but the Sum of the internal Angles = 360 or 2×180, therefore 180 : great Circle :: Sum of all the Angles of the Polygon — A (or $\overline{n-2} \times 180$) : Area of the Triangle.

SECT. IV.

The Solution of all the Caſes of ſpherical Triangles.

By help of the Propoſitions delivered in the laſt Section, all the Caſes of Spherical Triangles may be reſolved; I ſhall give the Solution of each Caſe in particular, both in right angled Triangles and oblique Triangles, after I have ſhown how they may all be reſolved in general by the foregoing Propoſitions. Every Spherical Triangle has 6 Parts, and any three being given the reſt may be found.

The Solution of right angled Triangles.

All the Caſes of right angled and quadrantal Triangles may be reſolved by the 26 and 27 Propoſitions; or by Prop. XXVIII alone.

1. The Solution of any Caſe may be performed by the 26 or 27 Prop. one of the two; either immediately in the Triangle it ſelf; or elſe in another right angled Triangle, form'd by producing all the Sides to Quadrants; wherein the Parts of the Triangle are either the ſame, or the Complements of thoſe in the firſt. Thus, let A B F be the propoſed Triangle, produce the Sides either thro' A or F, ſo that A C, A D, B E, and conſequently C E may be Quadrants. 41. Then you have two right angled Triangles A B F, E D F. In the Figure A B C D E F, mark what is given and its Complement with a Daſh (/); and what is ſought and its Complement with a Cypher (o). Then examine the two right Angled Triangles A B F, E D F; and obſerve in which of them, *an Angle and the Hypothenuſe, or an Angle and its oppoſite Side,* do

not

FIG. 41. not enter the Queſtion, (for there are always two parts out of the Queſtion); and that Triangle will afford the Solution, by applying Prop. XXVII. in the former Caſe, and Prop. XXVI. in the latter. If any ſide be greater than a Quadrant, then a Quadrant muſt be cut off, and the ſame Rule apply'd.

Quadrantal Triangles may be reſolved by the ſame Propoſitions, if you reduce them to right angled Triangles, by producing their oblique Sides to Quadrants, or cutting off a Quadrant if they be bigger. As in the Quadrantal Triangle A F E, produce A F, E F to Quadrants as A D, E B, and draw A B, E D round the Poles E, A, then you will have two right angled Triangles A B F, E D F, to be reſolved by the former Rule. Take an Example or two for right angled Triangles.

EXAMPLE I.

51. Let the Hypothenuſe AF and an Angle F be given, and the oppoſite Side A B required. I mark A F, F as given, and A B as ſought, then ſince there is an Angle A, and its oppoſite Side F B out of the Queſtion; therefore I need produce the Sides no further, for I have the Solution in the Triangle A B F, by Prop. XXVI. as Rad : S. A F : : S. F : S. A B required.

EXAMPLE II.

41. Let the Hyp. A F, Angle A be given; to find the other Angle F, I produce the Sides thro' F, and mark A F, F D, A, D C, E D as given; and F as ſought. Then I ſee that in the Triangle E F D, the Angle E and Hypothenuſe E F are not mark'd, and therefore by Prop. XXVII. S. F D : Rad : : Tan. E D : Tan. Angle F, that is Coſ. A F : Rad : : Cotan. A : Tan. F required.

2. Right angled or quadrantal Triangles may alſo be reſolved by Prop. XXVIII. Obſerving if mid-

dle Part be ſought, to begin with Radius; and if one extream be ſought, to begin with the other extream.

EXAMPLE I.

Given the Hyp. A F, and Angle F, to find the 51. Side A B; what is given and ſought being mark'd as in the Figure. Then AB is middle Part, and A F, F, extreams disjunct, therefore Rad: Coſ. Comp. A F:: Coſ. Comp. F : S.A B, that is Rad : S.A F :: S.F : S. A B ſought.

EXAMPLE II.

Given the Hyp. A F. and an Angle A, to find the 41. Angle F. here AF is middle Part, and A, F extreams conjunct. Therefore Tan. Comp. A : Rad :: S. Comp. A F : Tan. Comp. F, or Cotan. A : Rad :: Coſ. A F : Cotan. F.

EXAMPLE III.

In the quadrantal Triangle A F E, let Side A F, 41. Angle A, be given; to find the Angle F. Here A F is middle Part, and A and F are Extreams conjunct; therefore.

Tan. E A F : Rad :: Coſ. A F : Cotan, F.

Laſtly the Affection of the Angle or Side ſought will be known by Prop. XX. and Prop. XXI and its Corollaries; except it be ambiguous, and then it may be either acute or obtuſe.

The Solution of oblique ſpherical Triangles.

Oblique Triangles may be reſolved ſeveral ways, either by letting fall a Perpendicular, or without it. In any Triangle mark what is given with a daſh (/), and what is required with a Cypher (o), to diſtinguiſh them.

1. Oblique Triangles may be reſolved by the Rules of right angled Triangles, together with the Corollaries of Prop. XXVIII. with Prop. XXIX, XXXII. XXXIII, by letting fall a Perpendicular, which either divides the Triangle into two right angled Triangles; or makes two right angled Triangles, by adding a right angled Triangle to it.

In letting fall the Perpendicular, let it fall from the End of a given Side, and oppoſite to a given Angle; and if the three Things given be adjoining to one another, let it alſo fall from the End of the Side required, or oppoſite to the Angle required; and then you will have enough given in one of the right angled Triangles to find any of its unknown Parts: obſerving to find ſuch a Part (which muſt be either a Baſe or vertical Angle), as either of it ſelf, or when added to or ſubtracted from ſome Part given, the Proportion to find the Thing ſought may come under ſome of the Corollaries of Prop. XXVIII.

2. All the Caſes of oblique Triangles may alſo be reſolved by Prop. XXVII; XXIX, XXXII and XXXIII alone, by letting fall a Perpendicular according to the former Directions, when there is occaſion. For the two laſt Caſes will be reſolved by Prop. XXXIII, and XXXII, and all the reſt by Prop. XXVII and XXIX. and they are eaſily remembered, for three of them are the ſame with thoſe in plain Trigonometry, for reſolving the like Caſes. Only there will be more Operations in the right angled Triangles, than if the Corollaries of Prop. XXVIII were made uſe of.

3. All the Caſes of oblique Triangles (except the 1, 4, and 2 laſt) may alſo be reſolv'd by Prop. XXVIII, alone, by letting fall a Perpendicular dividing the whole into two right angled Triangles: In the firſt Triangle you have two Things given to find a Third, which muſt be either the Baſe or vertical Angle. In the ſecond Triangle you have one Thing given, or at leaſt you can find one Thing, after the firſt Opera-

FIG.

tion in the firſt Triangle is over; and in the ſecond Triangle there is alſo the Thing ſought. Then in theſe two Triangles you have either both the Baſes, or elſe both the vertical Angles. Therefore, from theſe, to find the Thing required, compare theſe three Things in the ſecond Triangle, *viz.* the Perpendicular, the Thing given, and the thing ſought, and find which is middle Part, and whether the other two be Extreams conjunct or disjunct, in order to know whether you muſt uſe Sines or Tangents. Do the ſame Thing in the firſt Triangle, with the three correſponding Parts; and (ſetting aſide the Perpendicular, which is an Extream in both Triangles;) then the middle Part and its Extream in the firſt Triangle, will be proportional to the middle Part and its extream in the ſecond Triangle.

53. *For Example.* Given FB, F and B; to find AF. In the firſt Triangle B F I, as Cotan. B : Rad : : Coſ. F B : Cotan. B F I. then you have A F I; then in the Triangle A F I, the Angle A F I is middle Part, and A F, F I Extreams conjunct. Likewiſe in the Triangle BFI; BFI, is middle Part, and B F, F I, Extreams conjunct. Therefore Coſ. B F I (middle Part) : Coſ. A F I (middle Part) : : Cotan. B F (Extream conjunct) : Cotan. A F (Extream conjunct).

The Reaſon of this Operation will appear thus, let p=Perpendicular, m=middle Part, a=the other Extream, in the firſt Triangle. M=middle Part, A= Extream in the ſecond Triangle. Then by Prop. XXVIII. $pa=mr$; and $pA=Mr$, therefore $\frac{p}{r}=\frac{m}{a}=\frac{M}{A}$ or $m:a::M:A$.

4. All oblique Triangles may alſo be reſolved without letting fall a Perpendicular, and theſe five Propoſitions, the XXIX, XXXVI, XXXVII, XL, XLI, and their Corollaries will reſolve them all. Thus, Caſe 1 and 4 are reſolved by Prop. XXIX. Caſe 2 and 6 by Prop. XXIX and Cor. 1. Prop. XXXVII.

Caſe

FIG. 53. Caſe 3 and 5 by Prop. XXIX, and Cor. 1. Prop. XXXVII. Caſe 7 by Prop. XXXVI, and Cor. 1. Caſe 8, by Prop. XXXVI, and Cor. 1. and Prop. XXIX. Caſe 9 by Prop. XXXVII, and Cor. 1. Caſe 10 by Prop. XXXVII, and Cor. 1. and Prop. XXIX. Caſe 11 by Prop. XLII, or its Corollaries. Caſe 12 by Prop. XLIII, or its Corollaries.

Laſtly the Affection of the Angle or Side ſought may be gathered from Prop. XVII, and its Corollaries. And the falling of the Perpendicular, from Prop. XIX. and Cor. Prop. VIII.

Here follow the 16 Caſes of right angled Triangles, and the 12 of oblique, In right angled Triangles, I ſhall give only the Solution by the Tables, omitting the Algebraic Solution; for that is very eaſily had by putting Letters inſtead of the Quantities, and transforming them by Sch. Prop. 1. I. if there is Occaſion.

In oblique Triangles you have both; where, note, that natural Sines, Tangents &c. muſt be uſed in the algebraic Way. I refer to the Caſes in right angled Triangles, in the Determination of the Species of the Angle or Side ſought, in the Method of letting fall a Perpendicular.

Right angled ſpherical Triangles.

CASE I.

Given the *Hypothenuſe* AF, and an *Angle* A; to find the *ſide Adjacent* A B, 52.

Rad : *Coſ.* Angle A : : *Tan. Hyp.* AF; *Tan.* adjacent Side A B.

If A, A F are of the ſame Affection, A B is leſs than a Quadrant; if of different Affection, more.

CASE II.

Given the *Hyp.* AF, and *Angle* A; to find the oppoſite *Side* B F. 52.

Rad : S. *Hyp.* AF : : S. *an Angle* A : S. *oppoſite Side* BF.

If A be acute, B F is leſs than 90; if obtuſe, more.

CASE III.

Given the *Hyp.* A F, and an *Angle* A; to find the other *Angle* F. 52.

Rad : *Coſ. Hyp.* A F : : *Tan. an Angle* A : *Cotan. other Angle* F.

When A F and A are of the ſame Affection, F is acute; if of different Affection, obtuſe.

CASE IV.

Given the *Hyp.* A F, and a *Leg* A B; to find the adjacent *Angle* A. 52.

Rad : *Cot. Hyp.* A F : : *Tan. a Side* A B : *Coſ. adjacent Angle* A.

If A B, A F be of the ſame Affection A is acute; if of different Affection, obtuſe.

CASE

FIG.

CASE V.

52. Given the *Hyp.* AF and one *Side* AB; to find the oppoſite *Angle* F.

S.*Hyp.* AF : *Rad* : : S. *one Side* AB : S. its *op.* *Angle* F.

If AB is leſs than 90, F is acute; if more than 90, obtuſe.

CASE VI.

52. Given the *Hyp.* AF and a *Leg* AB; to find the other *Leg* BF.

Coſ. one Leg AB : *Coſ. Hyp.* AF : : *Rad* : Coſ. *other Leg* BF.

If AB, AF be of one Affection, BF is leſs than 90; if of different Affection, more.

CASE VII.

52. Given a *Side* AB, and its adjacent *Angle* A; to find the oppoſite *Side* FB.

Rad : S. *one Side* AB : : *Tan. an Angle* A : *Tan. oppoſite Side* FB.

If A be acute, BF is leſs than 90; if obtuſe, greater.

CASE VIII.

52. Given a *Side* AB and its adjacent *Angle* A; to find the oppoſite *Angle* F.

Rad : S. *an Angle* A : : *Coſ. its adjacent Side* AB : *Coſ. its oppoſite Angle* F.

If AB is leſs than 90, F is acute; if more, obtuſe.

CASE

CASE IX.

Given a *Side* A B, and *Angle* adjacent A; to find the *Hyp.* A F. 52.

Coſ an Angle A : *Rad* : : *Tan. adjacent Side* A B : *Tan. Hyp.* A F.

If A and A B be of the ſame Affection, A F is leſs than 90; if different, more.

CASE X.

Given a *Side* B F, and oppoſite *Angle* A; to find the *Side* adjacent A B. 52.

Rad : *Cotan. an Angle* A : : *Tan. op. Side* B F : S. *adjacent Side* A B.

Here A B is ambiguous, that is, it may be either leſſer or greater than 90.

CASE XI.

Given a *Side* B F, and oppoſite *Angle* A; to find the *Angle* adjacent, F. 52.

Coſ. one Side B F : *Coſ. op. Angle* A : : *Rad* : S. *adjacent Angle* F. ambiguous.

CASE XII.

Given a *Side* B F, and *opp. Angle* A; to find the *Hyp.* A F. 52.

S. *an Angle* A : S. *op. Side* B F : : *Rad* : S. *Hyp.* AF, ambiguous.

CASE XIII.

Given the *Sides* A B, F B, to find an *Angle* A. 52.

S. *one Side* AB : *Rad* : : *Tan. other Side* B F : *Tan. op. Angle* A.

If BF is leſs than 90, A is acute; if more, obtuſe.

FIG.

CASE XIV.

52. Given the *Sides* A B, F B; to find the *Hyp.* A F.

Rad: *Coſ. one Leg* F B : : *Coſ. other Leg* A B : *Coſ. Hyp* A F.

If A B, F B are of one Affection, A F is leſs than a Quadrant; if of Different Affection, more.

CASE XV.

52. Given the *Angles* A, F; to find *a Side* B F.

S. *one Angle* F : *Rad* : : *Coſ. other Angle* A : *Coſ. op. Side* B F.

If A be acute, BF is leſs than a Quadrant; if obtuſe, it is greater.

CASE XVI.

52. Given the *Angles* A, F; to find the *Hyp.* A F.

Tan. one Angle F : *Cotan. other Angle* A : : *Rad* : *Coſ. Hyp.* A F.

If A, F be of the ſame Affection, A F is leſs than a Quadrant; if of different Affection, 'tis greater.

Oblique ſpherical Triangles.

CASE I.

Given two *Sides* A F, B F, and an oppoſite *Angle* A; to find the other oppoſite *Angle* B. 54. 55. 56.

I. *Logarithmically, by the Table of artificial Sines,* &c.

S. *one Side* F B : S. *op. Angle* A :: S. *other Side* A F : S. *its op. Angle* B.

Then $\frac{A+B}{2}$ is of the ſame Affection as $\frac{AF+BF}{2}$.

And if $\frac{A+B}{2}$ be of the ſame Affection as $\frac{A+\text{Sup. }B}{2}$, then B is ambiguous.

II. *Algebraically, by the Table of natural Sines,* &c.

Let $d=$S.AF, $p=$S.BF, $a=$S.A; then S.B$=\frac{ad}{p}$.

CASE II.

Given two *Sides* A F, B F, and an oppoſite *Angle* A; to find the included *Angle* F. 54. 55. 56.

I. *Logarithmically.*

By Caſe 3. *right* ∠ *Triangles,* R : *Coſ.* AF :: *Tan.* A : *Cotan,* A F I.

And *Cotan.* A F : *Coſ.* A F I :: *Cotan.*BF : *Coſ.* B F I.

Then according as B F and Angle A are of the ſame or different Affection, B F I is acute or obtuſe.

$$\left.\begin{array}{l}\text{If } AFI>BFI, \text{ then } AFI-BFI=F\\ \text{If } AFI+BFI<180, \text{ then } AFI+BFI=F\end{array}\right\}\text{ if both,}$$

then F is ambiguous.

FIG.

Or thus,

54. 55. Find the Angle B by Caſe 1. then

56. $S.\frac{AF \backsim BF}{2} : S.\frac{AF+BF}{2} :: Tan.\frac{B \backsim A}{2} : Cot.\frac{1}{2}F.$

II. *Algebraically.*

47. Let a, d, p be the *Sines*, and m, e, i the *Coſines* of A, AF, BF. And $b = \frac{ad}{p} =$ S.B, $n =$ Coſ. B. And $bb = bb - aa, = mm - nn, kk = 1 - bbpp = 1 - aadd$. Rad $= 1$.

Then $S.F = \frac{bb}{bim - aen} = \frac{bim + aen}{kk}$.

Alſo ſee Cor. 1, 2. Prop. XL.

CASE III.

54. 55. 56. Given two *Sides* AF, BF, and an oppoſite *Angle* A; to find the third *Side* AB.

I. *Logarithmically.*

By Caſe 1. *right* ∠ *Triangles,*

Rad : *Coſ.* A :: *Tan.* AF : *Tan.* AI.

Then *Coſ.* AF : *Coſ.* AI :: *Coſ.* BF : *Coſ.* BI.

Then if BF and A are of one Affection, BI is leſs than a Quadrant; if of different Affection, more.

$\left.\begin{array}{l}\text{If } AI > BI,\ AI - BI = AB, \\ \text{If } AI + BI < 180,\ AI + BI = AB,\end{array}\right\}$ if both, then AB is ambiguous.

Or thus.

Find the Angle B by Caſe 1. then

$S.\frac{B \backsim A}{2} : S.\frac{B+A}{2} :: Tan.\frac{AF \backsim BF}{2} : Tan.\frac{1}{2}AB.$

FIG.

II. *Algebraically.*

Let a, d, p be the *Sines*, and m, e, i the *Coſines* of A, AF, BF. $b = \frac{ad}{p} = \text{S.B}$, $n = $ *Coſ.* B, and $ll = dd - pp = ii - ee$, $tt = 1 - bbpp = 1 - aadd$, *Radius* $= 1$. Then 47.

$$\text{S.AB} = \frac{ll}{mid - pen} = \frac{mid + pen}{tt}.$$

See alſo Cor. 1, 2. Prop. XLI.

CASE IV.

Given two *Angles* A, B, and an oppoſite *Side* BF; to find the other oppoſite *Side* AF.

I. *Logarithmically.*

54.
55.
56.

S.A : S.BF :: S.B : S.AF.

$\frac{\text{AF}+\text{BF}}{2}$ is of the ſame Affection as $\frac{\text{A}+\text{B}}{2}$. If $\frac{\text{BF}+\text{AF}}{2}$ be of the ſame Affection as $\frac{\text{BF}+\text{Sup. AF}}{2}$, then AF is ambiguous.

II. *Algebraically.*

Let $a = \text{S.A}$, $b = \text{S.B}$, $p = \text{S.BF}$, then $\text{S.AF} = \frac{pb}{a}$.

CASE V.

Given two *Angles* A, B, and an oppoſite *Side* BF; to find the included *Side* AB. 54. 55. 56.

I. *Logarithmically.*

By *Caſe* 1. *right* ∠ *Triangles*, *Rad* : *Tan.* BF :: *Coſ.* B : *Tan.* BI.

Then *Cotan.* B : *Cotan.* A :: S.BI : S.AI.

FIG. If A, B be of the ſame Affection, $AI+BI=AB$.
54. If $BI+AI$ and $BI+$ Sup. AI be each <180, (AI
55. and conſequently) AB is ambiguous.
56. If A and B are of different Affection, $AI-BI=$
AB. If both AI and Sup. AI be $>BI$, then AI
and AB are ambiguous.

Or thus.

47. Find AF by Caſe 4. then $S.\frac{B\backsim A}{2} : S.\frac{B+A}{2} :: Tan.\frac{AF\backsim BF}{2} : Tan.\ \tfrac{1}{2}AB$.

II. *Algebraically.*

Let a, b, p be the *Sines*, m, n, i the *Coſines* of A, B, BF; $d=\frac{pb}{a}=S.AF$, $e=$ Coſ. AF, $ll=dd-pp=ii-ee$, $tt=1-bbpp=1-aadd$, Rad $=1$. Then

$$S.AB=\frac{ll}{mid-pen}=\frac{mid+pen}{tt}.$$

See alſo Cor. 1, 2, Prop. XLI.

CASE VI.

54. Given two *Angles* A and B, and an oppoſite *Side*
55. BF: to find the third *Angle* F.
56.

I. *Logarithmically.*

(By *Caſe* 3 *right* ∠'*s*) *Rad*: *Tan*: B :: *Coſ.* BF: *Cotan.* BFI;

Then *Coſ.* B : S. BFI : *Coſ.* A : S. AFI.

If A and B are of the ſame Affection, $BFI+AFI=F$. If both $BFI+AFI$ and $BFI+$Sup. AFI be <180, then (AFI and) F is ambiguous.

If A and B are of different Affection, $AFI-BFI=F$, if both AFI and Sup. $AFI>BFI$, then (AFI and) F is ambiguous.

Or thus. FIG.

Find A F by Caſe 4. then S. $\frac{AF \backsim BF}{2}$: 54.
S. $\frac{AF+BF}{2}$: : *Tan.* $\frac{B \backsim A}{2}$: *Cotan.* $\frac{1}{2}$ F. 55. 56.

II. *Algebraically.*

Let a, b, p be the *Sines*; m, n, i the *Coſines* of A, B, BF; $d=\frac{pb}{a}=$ S.AF, $e=$ *Coſ.* AF. $bb=bb-aa=mm-nn$, $kk=1-bbpp=1-aadd$, Rad$=1$. Then 47.

S.F$=\frac{bb}{bim-aen}=\frac{bim+aen}{kk}$.

Alſo ſee Cor. 1, 2. Prop. XL.

CASE VII.

Given two *Sides* A F, A B, and the included *Angle* A; to find an oppoſite *Angle* B. 54. 55. 56.

I. *Logarithmically.*

(By Caſe 1 right $\angle$'s) *Rad*: Coſ. A : : *Tan.* A F : *Tan.* A I, and A B ∽ A I$=$I B.

Then S. A I : S. B I : : *Cotan.* A : *Cotan.* B.

According as A I is $<$ or $>$ than A B, B and A are of the ſame or different Affection.

Or thus.

Coſ. $\frac{AB+AF}{2}$: *Coſ.* $\frac{AB \backsim AF}{2}$: : *Cotan.* $\frac{1}{2}$ A : *Tan.* $\frac{F+B}{2}$.

And S. $\frac{AB+AF}{2}$: S. $\frac{AF \backsim AB}{2}$: : *Cotan.* $\frac{1}{2}$ A : *Tan.* $\frac{F \backsim B}{2}$.

Then

FIG. 54. 55. 56. Then $\frac{F+B}{2}+\frac{F\backsim B}{2}=$ *Angle oppoſite* to the greater Side.

And $\frac{F+B}{2}-\frac{F\backsim B}{2}=$ *Angle op.* to the leſſer Side.

Note, $\frac{F+B}{2}$ is of the ſame Affection as $\frac{AF+AB}{2}$.

II. *Algebraically.*

47. Let s, a, d be the *Sines*, c, m, e the *Coſines* of A B, A, A F. $z=$ *Cotan.* A B, $t=$ *Cotan.* A F, $\tau=$ *Cotan.* A, *Rad*$=1$. Then

Cotan. $B=\frac{se-dcm}{da}=\frac{s}{a}t-c\tau$.

Cotan. $F=\frac{dc-sem}{as}=\frac{d}{a}z-e\tau$.

CASE VIII.

54. 55. 56. Given two *Sides* A F, A B, and the included *Angle* A; to find the third Side F B.

I. *Logarithmically.*

(By *Caſe* 1 *right* ∠*'s*) Rad: *Coſ.* A :: *Tan.* A F : *Tan.* A I, and A B ∽ A I = B I.

Then *Coſ.* A I : *Coſ.* A F :: *Coſ.* B I : *Coſ.* B F.

As B I and A are of the ſame or different Affection, B F is < or > than a Quadrant.

Or thus

Find the ∠ B by Caſe 7, then S. B ; S. A F :: S. A : S. B F, by Caſe 4.

II. *Algebraically.*

47. Let s, d be the *Sines*, and c, e the *Coſines* of A B, A F; $m=$ *Coſ.* A, $v=$ *verſ.* A. D = Coſ. $\overline{AB \backsim AF}$. *Rad*$=1$. Then *Coſ.* F B $=sdm+ce=D-sdv$.

CASE IX.

FIG.

Given two *Angles* A and F, and the included *Side* A F; to find an oppoſite *Side* F B. 54. 55. 56.

I. *Logarithmically.*

By *Caſe* 3 *right* ∠'s, *Rad*: *Tan.* A :: *Coſ.* A F : *Cotan.* A F I. and A F B ∽ A F I = B F I; then,

Coſ. A F I : *Coſ.* B F I :: *Cotan.* A F : *Cotan.* B F. as B F I and A are of the ſame or different Affection, B F is < or > than a Quadrant.

Or thus.

$$\text{Coſ.}\,\frac{F+A}{2} : \text{Coſ.}\,\frac{F \backsim A}{2} :: \text{Tan.}\,\tfrac{1}{2}AF : \text{Tan.}\,\frac{AB+BF}{2},$$

$$\text{And S.}\,\frac{F+A}{2} : \text{S.}\,\frac{F \backsim A}{2} :: \text{Tan.}\,\tfrac{1}{2}AF : \text{Tan.}\,\frac{AB \backsim BF}{2}.$$

Then $\frac{AB+BF}{2} + \frac{AB \backsim BF}{2} =$ *Side* oppoſite to the greater Angle.

And $\frac{AB+BF}{2} - \frac{AB \backsim BF}{2} =$ *Side* opp. to the leſſer Angle.

Note, $\frac{AF+BF}{2}$ is of the ſame Affection as $\frac{F+A}{2}$.

II. *Algebraically.*

Let a, d, f be the *Sines*, and m, e, g the *Coſines* of A, A F and F, $\tau =$ *Cotan.* A, $t =$ *Cotan.* A F, $y =$ *Cotan.* F, *Radius* $= 1$, then 47.

$$\text{Cotan. BF} = \frac{f\tau + ge}{d} = \frac{f\tau}{d} + gt:$$

$$\text{And Cotan. AB} = \frac{ay+me}{d} = \frac{ay}{d} + mt.$$

FIG.

CASE X.

54.

55. Two *Angles* A and F, and the included *Side* A F,
56. being given; to find the third *Angle* B.

I. *Logarithmically*.

By *Caſe* 3. *right* ∠'s. *Rad* : *Tan.* A :: *Coſ.* A F : *Cotan.* A F I, and $AFB \backsim AFI = BFI$, then

S. A F I : S. B F I :: *Coſ.* A : *Coſ.* B.

As A F I is $<$ or $>$ than A F B, B and A are of the ſame or different Affection.

Otherwiſe.

Find FB by Caſe 9, then B by Caſe 1.

II. *Algebraically*.

47. Let a, f be the *Sines*, and m, g the *Coſines* of A and F, $e =$ *Coſ.* A F, *Rad* $= 1$, then

Coſ. $B = afe - mg$. Alſo ſee Prop. XLIII.

CASE XI.

54. Given the three *Sides*, to find an *Angle* A.

55.

I. *Logarithmically*.

56.

$$\textit{Tan.}\ \tfrac{1}{2}AB : \textit{Tan.}\ \frac{AF+FB}{2} :: \textit{Tan.}\ \frac{AF \backsim FB}{2} :$$

Tan. an Arch Q. If $\tfrac{1}{2}AB > Q$, the Perpendicular falls within; if leſs, without.

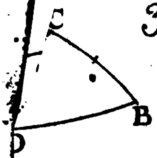
Then $\tfrac{1}{2}AB + Q =$ greater Segment A I or B R, and $\tfrac{1}{2}AB \backsim Q =$ leſſer Segment B I or A R. then

If $AF+FB < 180$, the Perpendicular falls neareſt the leſſer Side.

If $AF+FB > 180$, the Perpendicular falls neareſt the greater Side.

Then by *Caſe* 4. *right* ∠'s. *Rad* : *Tan.* AI :: *Cotan.* A F : *Coſ.* A.

Or



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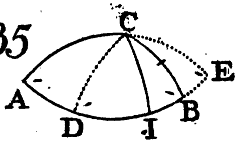

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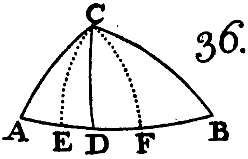

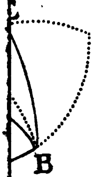

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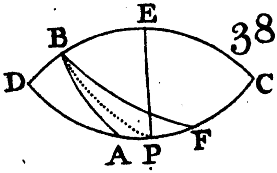

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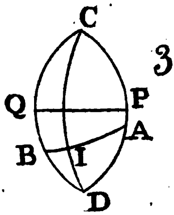

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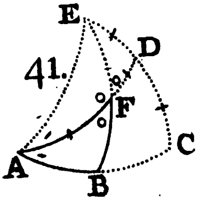

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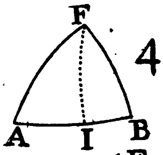

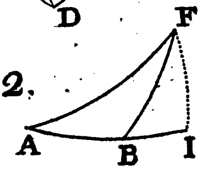

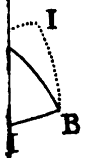

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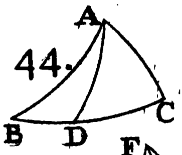

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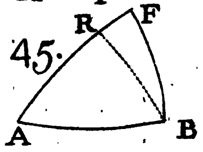

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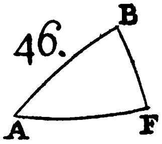

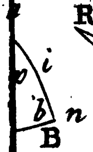

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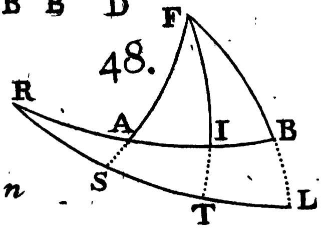

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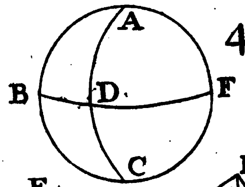

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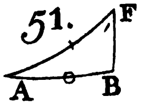

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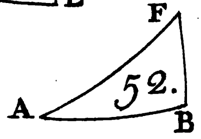

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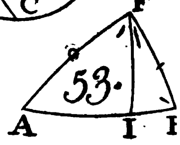

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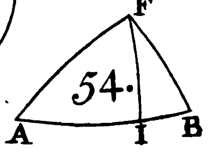

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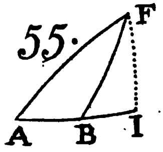

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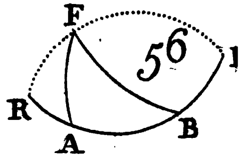

VI. *the End:*

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FIG.

Or thus,

$S.AB\times S.AF$: *Rad Square* :: $S.\frac{BF+AB-AF}{2}\times$ 54. 55. 56.

$S.\frac{BF+AF-AB}{2}$: S. Square $\frac{1}{2}$ A.

Or $S.AB\times S.AF$: *Rad. Square* :: $S.\frac{BF+AB+AF}{2}$

$\times S.\frac{BF+AB+AF}{2}-BF$: *Coſ. Square* $\frac{1}{2}$ A.

See alſo Prop. XLII and Cor.

II. *Algebraically.*

Let d, s be the *Sines*, and e, c the *Coſines* of AF, AB; $i=$ *Coſ.* FB, $z=$ *Coſ.* $\overline{AB+AF}$, $x=$ *Coſ.* $\overline{AB\backsim AF}$, *Radius* $=1$, then *Coſ.* $A=\frac{i-ec}{ds}$, 47.

and *verſ.* $A=\frac{x-i}{x-z}\times 2$.

CASE XII.

The three *Angles* being given; to find a *Side* A F. 54. 55. 56.

I. *Logarithmically.*

As A, B are of the ſame or different Affection; the Perpendicular falls within or without.

Cot. $\frac{A+B}{2}$: *Tan.* $\frac{A\backsim B}{2}$ :: *Tan.* $\frac{1}{2}$F : *Tan.* an Arch Q.

Then $\frac{1}{2}F+Q=$ greater Angle at the Vertex AFI or BFR.

And $\frac{1}{2}F\backsim Q=$ leſſer Angle at the Vertex BFI or AFR.

Then if $A+B<180$, the Perpendicular falls neareſt the leſſer Side or greater Angle.

If $A+B>180$, the Perpendicular falls neareſt the greater Side or leſſer Angle.

Then by Caſe 16. *right Angles*, *Tan.* AFI : *Cotan.* A :: *Rad.* : *Coſ.* AF.